Math 341 – Homework #5
Due Wednesday, March 4, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6, 2.1, and 3.4 of our textbook.

1. Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 7 \\ 1 & 2 & 8 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$, and let $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation associated to A, that is
$$L_A(x) = Ax.$$
   (a) Find a basis for the image of the linear transformation associated to the matrix . (This is the matrix from the first problem of Worksheet #4.)

   (Hint: You know that the image of a basis for the domain will be a spanning set for the image (Theorem 2.2). Use the standard basis for the domain, and come up with a spanning set for the image. Is it linearly independent? If not... see Problem 1 on the previous homework assignment.)

   (b) Find the general solution to the homogeneous equation $Ax = 0$, and give a basis for the kernel of L_A.

2. Give the general solution to the following system of equations.
$$\begin{aligned} 2x_1 + x_3 - 4x_5 &= 5 \\ 3x_1 - 4x_2 + 8x_3 + 3x_4 &= 8 \\ x_1 - x_2 + 2x_3 + x_4 - x_5 &= 2 \\ -2x_1 + 5x_2 - 9x_3 - 3x_4 - 5x_5 &= -8 \end{aligned}$$

3. The system of equations from the previous problem can be expressed as the single equation $Bx = \begin{pmatrix} 5 \\ 8 \\ 2 \\ -8 \end{pmatrix}$, for some matrix B, where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$. Let L_B be the linear transformation associated to B. What is the domain of L_B? What are the rank and nullity of L_B? Give a basis for both the image and the kernel of L_B.

4. Suppose that the matrix M' can be obtained from the matrix M by elementary row operations. Prove that M can be obtained from M' by elementary row operations.

   (Hint: first consider the case where only one operation is used, and consider each type seperately.)

5. Suppose that the matrix P' can be obtained from the matrix P by performing elementary row operations.

   (a) Prove that $\ker(L_P) = \ker(L_{P'})$.

   Hint: You can use the result of Problem 4.

(b) Give an example that shows that the images of L_P and $L_{P'}$ may not be equal.

Hint: Remember that the columns of a matrix span the image of the associated linear transformation. For your example, you will need to have an $m \times n$ matrix (you choose the m and n) where the columns do not span F^m (why?)

6. Suppose that C is a 3×3 matrix, and that we can obtain the 3×3 identity matrix $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ from C by performing elementary row operations. Prove that the associated linear transformation $L_C : F^3 \to F^3$ is one-to-one and onto.