Math 341 – Homework #4
Due Wednesday, February 25, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6 and 2.1 of our textbook.

1. Define a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(a, b, c) = (a - b + c, 3a + c, 5a + b + c)$.

   (a) Prove that T is linear.

   (b) Compute the image of T as follows. First describe a basis $\{v_1, v_2, v_3\}$ for the domain. By a theorem we have discussed, the set $\{T(v_1), T(v_2), T(v_3)\}$ spans $\operatorname{im}(T)$. Check to see if these vectors are linearly independent. If they are, they form a basis. Otherwise, one of the vectors is in the span of the other two, so it may be removed from the set without reducing the span. Keep repeating this procedure until you have a linearly independent set.

   (c) What is the dimension of $\ker(T)$?

2. Define a transformation $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^2$ by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a - b + c, 3a + d)$.

   (a) Prove that T is linear

   (b) Describe the kernel and image of T.

   (c) Is T one-to-one? Is T onto? Explain.

3. Example 2 on page 66 of our textbook describes the function $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ which is a rotation of the plane by the angle θ. Read through this example.

   (a) The authors say it's easy to show T_θ is a linear transformation (for any θ). Prove this.

   (b) For any θ, prove that T_θ is one-to-one (and therefore also onto).

4. Let V and W be vector spaces over a field F, and let $T : V \to W$ be a linear transformation. Prove the following two claims:

   (a) If $\dim(V) > \dim(W)$, then T is not one-to-one.

   (b) If $\dim(V) < \dim(W)$, then T is not onto.

5. Let V and W be vector spaces over a field F. Let V' be a subset of V and let W' be a subset of W. Let $T : V \to W$ be a linear transformation.

   (a) We define the *image of V' under T*, denoted $T(V')$, to be

   $$T(V') = \{T(x) \mid x \in V'\}.$$

   In other words, $T(V')$ is the set of all vectors which are outputs corresponding to inputs from the subspace V'. Prove that if V' is a subspace of V, then $T(V')$ is a subspace of W.

(b) We define the *preimage of W' under T*, denoted $T^{-1}(W')$, by

$$T^{-1}(W') = \{x \in V \mid T(x) \in W'\}.$$

In other words, $T^{-1}(W')$ is the set of all inputs whose output lies in the subspace W'. Prove that if W' is a subspace of W, then $T^{-1}(W')$ is a subspace of V.

Note: the notation $T^{-1}(\ \ )$ can be confusing at first, because growing up we only use the notation T^{-1} to denote the inverse of a function. We can use this notation *even if T does not have an inverse*, that is, even if T is not one-to-one. If T is one-to-one, then for each $y \in \operatorname{im}(T)$, the set $T^{-1}(\{y\})$ constists of a single point. If T is not one-to-one, then $T^{-1}(\{y\})$ may consist of many points.