Math 341 – Homework #3
Due Wednesday, February 18, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6 and 2.1 of our textbook. Specifically important will be the following:

- The *nullity* of a linear transformation T, denoted $\text{nullity}(T)$, is the dimension of $\ker(T)$. The *rank* of T, denoted $\text{rank}(T)$, is the dimension of $\text{im}(T)$. (You will have proved on Worksheet #3 that $\ker(T)$ and $\text{im}(T)$ are vector spaces, so that it makes sense to talk about their dimensions.)
- The Rank-Nullity Theorem (called the "Dimension Theorem" in our textbook) states that if V and W are vector spaces, and if $T : V \to W$ is a linear transformation, then

$$\dim(V) = \text{rank}(T) + \text{nullity}(T).$$

  We will discuss this theorem in class on Monday, 2/16, but in the mean time, you may take it for granted.

1. Define a function $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(a_1, a_2, a_3) = (a_1 + a_2 + a_3, 2a_1 + 2a_2 + 2a_3).$$

   (a) Prove that T is a linear transformation.
   (b) Describe $\ker(T)$ and $\text{im}(T)$ geometrically (i.e. in terms of lines, planes, etc... can you draw pictures of them?)
   (c) Find a basis for $\ker(T)$ and a basis for $\text{im}(T)$.
   (d) Verify that $\text{rank}(T) + \text{nullity}(T) = 3$.

2. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$. This is what you might have called a "linear function" earlier in life. Prove that f is a linear transformation from $\mathbb{R}^1$ to $\mathbb{R}^1$ if and only if $b = 0$. (That is, if and only if the line passes through the origin.)

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation, and assume that $T((1, 2)) = (1, 1, 0)$ and $T((-1, 1)) = (0, 1, 1)$. What is $T((3, 4))$? Explain. (One way to do this problem is to find a formula for such a linear transformation, but you actually do not need to do this to solve the problem.)

4. This problem is a follow-up on Problem 5 from Homework #2. Let Z be the space of trace-zero matrices in $M_{n\times n}(F)$. In that problem you (should have) found that $\dim(Z) = n^2 - 1$ by constructing a basis for Z with $n^2 - 1$ vectors. In the following steps you will prove that $\dim(Z) = n^2 - 1$ by a different argument.

   (a) Prove that the trace function $\text{tr} : M_{n\times n}(F) \to F^1$ is a linear transformation. (We basically did this on Worksheet #1 already, but we weren't using the words "linear transformation" at the time.)
   (b) Explain what $\ker(\text{tr})$ and $\text{im}(\text{tr})$ are.
   (c) State the dimension of $M_{n\times n}(F)$ and the dimension of $\text{im}(tr)$ (i.e. the rank of the trace function).

(d) Use the Rank-Nullity Theorem to conclude immediately that $\dim(Z) = n^2 - 1$.

5. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Let $v_1, \ldots, v_n$ be vectors in V. Prove that if the vectors $T(v_1), \ldots, T(v_n)$ form a linearly independent subset of W, then $\{v_1, \ldots, v_n\}$ is a linearly independent subset of V.