Math 341 – Homework #2
Due Wednesday, February 11, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6 of our textbook.

1. For each of the following, state (with proof) whether or not the set is a basis for $\mathbb{R}^3$.
   (a) $S = \{(1,2,3), (-1,3,2), (0,1,-1)\}$
   (b) $T = \{(1,0,1), (1,1,0)\}$
   (c) $B = \{(1,2,3), (-2,-6,-4), (0,1,-1)\}$
   (d) $C = \{(1,2,0), (1,0,0), (0,1,-1), (0,0,2)\}$

2. Let V be a vector space over a field F. Let $S = \{u_1, \dots, u_n\}$ be a set of vectors in V. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{Span}(\{u_1, u_2, \dots, u_k\})$ for some k with $1 \leq k < n$.

3. Let V be a vector space over a field F, and let S and T be subsets of V. Prove that $\text{Span}(S \cap T) \subseteq \text{Span}(S) \cap \text{Span}(T)$. Give an example which shows that $\text{Span}(S \cap T)$ can be a proper subset of $\text{Span}(S) \cap \text{Span}(T)$.

4. An $n \times n$ matrix is called *diagonal* if its only non-zero entries are "on the diagonal." That is, A is diagonal if $A_{ij} = 0$ whenever $i \neq j$. The set D of all $n \times n$ diagonal matrices is a subspace of $M_{n \times n}$. What us the dimension of D? (Exhibit a basis.)

5. On Worksheet #1 we proved that the set Z of trace-zero matrices is a subspace of $M_{n \times n}(F)$. What is the dimension of Z? (Exhibit a basis.)

6. Let V be a finite-dimensional vector space with two subspaces W_1 and W_2. Prove that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

   Recall (from Worksheet #1) that $W_1 + W_2 = \{x + y \mid x \in W_1 \text{ and } y \in W_2\}$.

   (Hint: start with a basis $\mathcal{B}$ for $W_1 \cap W_2$. You can extend $\mathcal{B}$ to a basis for W_1, and you can also extend $\mathcal{B}$ to a basis for W_2. Try to write down a basis for $W_1 + W_2$ in terms of the vectors that make up these bases.)