Math 341 – Homework #12
Due Wednesday, April 29, 2015

We'll begin with some definitions. The complex numbers, denoted $\mathbb{C}$, are defined as the set of all numbers of the form $a + bi$, where a and b are real numbers. A real number is also a complex number (with $b = 0$.) The "imaginary" number i satisfies the condition that $i^2 = -1$. We add two complex numbers in the usual way:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and we multiply using the FOIL rule, and simplify:

$$(a + bi)(c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (ad + bc)i.$$

One can easily check that, with these two operations, $\mathbb{C}$ is a field.

Let $z = a + bi$ be a complex number, where a and b are real numbers.

- The number a is called the *real part* of z, denoted $\Re(z)$, and b is called the *imaginary part* of z, denoted $\Im(z)$. ($\Im$ is actually a funny looking "I." This font and notation are fairly standard.) Notice that the real and imaginary parts of z are real numbers. For example if $z = 3 - 4i$, then $\Re(z) = 3$ and $\Im(z) = -4$.
- We define the *absolute value* (sometimes called the *modulus*) of a complex number z, denoted $|z|$, by
$$|z| = \sqrt{a^2 + b^2}.$$
- We define the *complex conjugate* of z, denoted $\overline{z}$, by
$$\overline{z} = a - bi.$$
For example, if $z = 3 - 5i$, then $\overline{z} = 3 + 5i$. Notice that we always have $\overline{\overline{z}} = z$. Also note that $\overline{z} = z$ if and only if z is real, and $\overline{z} = -z$ if and only if $\Re(z) = 0$. (If $\Re(z) = 0$ and $z \neq 0$, we say z is *purely imaginary*.)

1. If z is a complex number, prove that $z\overline{z} = |z|^2$.
2. The only interesting part of checking that $\mathbb{C}$ is a field is the finding of multiplicative inverses. Find the multiplicative inverse of $2 + i$. (We would denote this number as $\frac{1}{2+i}$.
3. Find the multiplicative inverse z^{-1} of a generic complex number $z = a + bi$ (assuming a and b aren't both zero). You should show clearly how you figured out what z^{-1} was, then show how you verify that $zz^{-1} = 1$.
4. Show that complex conjugation distributes over all usual operations, that is, for any $w, z \in \mathbb{C}$ we have
   (i) $\overline{w \pm z} = \overline{w} \pm \overline{z}$,
   (ii) $\overline{wz} = \overline{w} \cdot \overline{z}$, and
   (iii) $\overline{(w/z)} = \overline{w}/\overline{z}$ (as long as $z \neq 0$).

5. What is i^{2015}? (Explain very briefly.)

6. What is $\Re(\frac{5}{-3+4i})$?

7. Find all complex solutions to the equation $z^4 = 1$. (There are 4 solutions†. No need to show work here.)

8. Show that $\left(\frac{1+i\sqrt{3}}{2}\right)^6 = 1$. The equation $z^6 = 1$ has 6 solutions in $\mathbb{C}$. Can you find them all?

9. Let $z = a + bi$, where a and b are real numbers. Prove that $\Re(z) = \frac{1}{2}(z + \overline{z})$ and $\Im(z) = \frac{1}{2i}(z - \overline{z})$.

10. If $w, z \in \mathbb{C}$ and either $|w| = 1$ or $|z| = 1$, prove that $\left|\frac{w-z}{1-\overline{w}z}\right| = 1$ (as long as the denominator is not 0).

    Hint: Use Problems 1 and 3.

†Solutions to an equation of the form $z^n = 1$ are called *roots of unity*.