Math 341 – Homework #11
Due Wednesday, April 22, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2-1.6, 2.1-2.5, 3.1-3.4, 4.1-4.4, and 5.1-5.2 of our textbook, as well as notes from lecture.

1. Let $A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A, and say what the eigenvalues are, and what the (algebraic) multiplicity of each eigenvalue is.

   (b) Find a basis for the eigenspace associated to each eigenvalue of A.

   (c) Is A diagonalizable? Explain why or why not.

2. If A is a diagonalizable matrix, prove that A^t is also diagonalizable. Hint: in a previous homework, you should have proved that $(A^t)^{-1} = (A^{-1})^t$. You may use this fact freely.

3. Let V is a vector space and $T \in \mathcal{L}(V)$. If W is a subspace of V, we say that W is *T-invariant* if $T(W) \subseteq W$. That is, W is T-invariant if for every $x \in W$ we have $T(x) \in W$.

   (a) For any linear operator $T \in \mathcal{L}(V)$, prove that $\ker(T)$ is T-invariant.

   (b) If $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T, then prove that the eigenspace E_λ is T-invariant.

   (c) If W is a T-invariant subspace of V and W' is a subspace of W, must W' also be T-invariant? If so, prove it. If not, give an example that shows this.

4. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

   (a) Write down the characteristic polynomial of A, and "expand it all the way," i.e. write it out with no parentheses. Call the characteristic polynomial $f(t)$.

   (b) Write down what A^0, A^1, A^2, and A^3 are. To explain the notation, we define A^0 to be the identity matrix, $A^1 = A$, $A^2 = AA$, etc. (This is what a power of a matrix means – it's like regular powers, but we're using matrix multiplication.)

   (c) Show that $f(A) = 0$.
   (What does this mean? It's a matrix equation. $f(A)$ means you "plug A into the polynomial. You should interpret powers of A as in part (b), and interpret the constant term as a scalar matrix, i.e. a scalar times the identity matrix. The 0 on the right-hand-side represents the zero matrix. You are showing that A "satisfies its own characteristic equation." Here the characteristic equation of a matrix (or linear operator) means the equation $f(t) = 0$, where $f(t)$ is the characteristic polynomial.)

   (d) Draw a picture of how this makes you feel.