Math 341 – Homework #1
Due Wednesday, February 4, 2015

For this assignment, you should refer to the definitions and theorems of Sections 1.2 and 1.3 of our textbook.

1. Let V be a vector space over a field F. Prove the "FOIL" rule:

   for all a, b in F and x, y in V,

   $$(a+b)(x+y) = ax + ay + bx + by.$$

2. Prove Corollary 2 of Theorem 1.1 from our text: the vector y described in (VS4) is unique. (In other words, a vector can only have one additive inverse.)

3. Let V be the set of ordered pairs (a_1, a_2) of real numbers, with addition and scalar multiplication given by the following:

   $$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2), \text{ and} \qquad (1)$$
   $$c(a_1, a_2) = (ca_1, a_2). \qquad (2)$$

   For each of the 8 vector space axioms, explain why the axiom is or is not satisfied. Is V a vector space over $\mathbb{R}$?

4. In class we showed that, for any set S and field F, the set of functions from S to F, denoted $\mathcal{F}(S, F)$, is a vector space over F (cf. Example 3 from Sec. 1.2 of our text). Prove that $\mathcal{C}(\mathbb{R}, \mathbb{R})$, the set of *continuous* functions from $\mathbb{R}$ to $\mathbb{R}$ (like those studied in calculus), is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. You may assume without proof whatever basic theorems from calculus about continuous functions you like.

Definition. Let V be a vector space, and let W_1 and W_2 be subspaces. We say that V is the *direct sum* of W_1 and W_2 if

(a) $W_1 \cap W_2 = \{0\}$, and

(b) $V = W_1 + W_2$.

We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$.

5. Prove that

   $$\mathbb{R}^2 = W_1 \oplus W_2,$$

   where $W_1 = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$, and $W_2 = \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$.

6. In Example 4 (and the preceding paragraphs) of Sec. 1.2 of our text, the authors introduce the vector space $P(F)$ of polynomials with coefficients in $F^\dagger$. Let

   $$E(F) = \{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in P(F) \mid a_i = 0 \text{ whenever } i \text{ is odd}\},$$

   that is, E is the set of all polynomials whose only non-zero coefficients are those corresponding to even powers of x. For example, $x^8 + 2x^2 - 4 \in E(\mathbb{R})$. Similarly, let

   $$O(F) = \{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in P(F) \mid a_i = 0 \text{ whenever } i \text{ is even}\}.$$

   Prove that $P(F) = E(F) \oplus O(F)$.

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†N.B. A more common notation (outside of our textbook) for $P(F)$ is $F[x]$, where x is the "indeterminate variable."

7. A matrix A is called *symmetric* if $A^t = A$, and A is called *skew-symmetric* if $A^t = -A$. Here the notation A^t denotes the *transpose* of the matrix A, which is the matrix obtained by exchanging rows for columns, and vice versa (see page 17 of the text for more on transposes).

   For example, considering 3×3 matrices with real entries, the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & -3 \end{pmatrix}$ is symmetric, while $\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & -1 \\ -3 & 1 & 0 \end{pmatrix}$ is skew-symmetric. (Notice that only square matrices can be symmetric or skew-symmetric, and also that a skew-symmetric matrix must have all zeros on the main diagonal.) Prove that $M_{3\times 3}(\mathbb{R}) = S \oplus S'$, where S denotes the set of 3×3 symmetric matrices, and S' denotes the set of 3×3 skew-symmetric matrices. You may assume without proof that S and S' are subspaces of $M_{3\times 3}(\mathbb{R})$. (If you want more practice, go ahead and prove S and S' are subspaces first. The symmetric case is proved starting on page 17 of our text. Once you write your proof that $M_{3\times 3}(\mathbb{R})$, you'll see it would have been just as easy to prove this statement for general $n \times n$ matrices – if you want to do this for fun, go ahead. You can also replace $\mathbb{R}$ with any field where you can "divide by 2.")