HAMILTONIAN EVOLUTIONS OF CURVES IN CLASSICAL AFFINE GEOMETRIES

GLORIA MARÍ BEFFA

Abstract. In this paper we study geometric Poisson brackets and we show that, if $M = (G \ltimes \mathbb{R}^n)/G$ endowed with an affine geometry (in the Klein sense), and if G is a classical Lie group, then the geometric Poisson bracket for parametrized curves is a trivial extension of the one for unparametrized curves, except for the case $G = \mathrm{GL}(n, \mathbb{R})$. This trivial extension does not exist in other nonaffine cases (projective, conformal, etc).

1. Introduction

In the last years a large number of papers have appeared in the literature linking invariant evolutions of curves to completely integrable systems, much like the Vortex Filament flow is related to the non-linear Schrödinger equation via the Hasimoto transformation. Other examples can be found in [1, 2, 5, 6, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 26, 27, 28, 29]. An interesting difference between the many examples is that some of them correspond to arc-length preserving evolutions (for example Vortex filament), while others do not preserve arc-length or any other invariant parameter (for example the KdV Schwarzian evolution).

In several recent papers ([15]-[21]) the author studied this problem from the point of view of biHamiltonian structures. She defined geometric Poisson brackets as brackets defined on the space of differential invariant of curves through reduction of well-known Poisson bracket in the algebra of Loops on a Lie algebra. These structures seemed to exist in pairs whenever a geometric realization of an integrable system was present, but one of them always existed by itself, for any geometric setting. Most known biHamiltonian structures for completely integrable PDEs are geometric Poisson brackets. The construction of these brackets would require the preservation of arc-length in some cases (for example the Euclidean case, where the Vortex Filament flow exists), while other cases did not require it (for example RP^1 where the Schwarzian KdV flow exists). The question of whether or not the geometry itself imposes the preservation condition on the flow has not been resolved yet. In this note we aim to clarify the situation further.

Let $M = (G \ltimes \mathbb{R}^n)/G \cong \mathbb{R}^n$ be a manifold endowed with an affine geometry (in the Klein sense) determined by the affine action of $G \ltimes \mathbb{R}^n$ on $\mathbb{R}^n$ given by $(g, v) \cdot u = gu + v$. Assume $G \subset \mathrm{GL}(n, \mathbb{R})$ is semisimple. Associated to M there is a natural Hamiltonian structure $\{,\}(\mathbf{k})$ defined on the space of differential invariants of parametrized curves with a monodromy in the group, i.e., curves $u(x) \in \mathbb{R}^n$

Date: April 14, 2008.

1991 *Mathematics Subject Classification.* Primary: 37K; Secondary: 53A55.

Key words and phrases. completely integrable evolutions, biHamiltonian structures, affine geometries.

such that $u(x+T) = mu(x)$ for all x, where $m \in G$ and $T \in \mathbb{R}$ is the period (these curves have periodic differential invariants, other conditions can be imposed to ensure that the coefficients vanish at infinity, for example). The author showed in [15] that, locally on the neighborhood of a nondegenerate curve, the space of differential invariants of curves could be written as a quotient of the form $U/\mathcal{L}N$ where $U \subset \mathcal{L}\mathfrak{g}^* = C^\infty(S^1, \mathfrak{g}^*)$ is an open subset and $N \subset G$ is the isotropy subgroup of a certain element $\Lambda \in \mathbb{R}^n$. This fact led to the definition of a natural Poisson structure on the space of differential invariants of parametrized curves, obtained by reduction from a natural and well-known Poisson bracket on $\mathcal{L}\mathfrak{g}^*$.

One of the generators of the manifold of differential invariants is the invariant used to define arc-length (and other similar ones), for example $k = u_1 \cdot u_1$ in the case of Euclidean geometry $G = O(n)$, $k = u_1^T J u_2$ (J= $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the symplectic matrix), in the case of symplectic geometry $G = Sp(n)$, $k = \det(u_1, \dots, u_n)$ in the case of equiaffine geometry $G = SL(n)$, etc. An appropriate power of each one of these lowest order invariants can be used to define an invariant one form, and hence a special choice of parameter that, because of its parallelism with Euclidean geometry, we will call parameter of arc-length type. Its associated differential invariant will be called *invariant of arc-length type*.

In this note we will investigate two points; first of all we will look into properties of those evolutions of curves on the manifold that induce a Hamiltonian evolution on the invariants of the flow. In particular, we prove the following fact: consider $G \subset \mathrm{GL}(n)$ to be a classical Lie group. Let

$$u_t = F(u, u_x, u_{xx}, \dots) \tag{1.1}$$

be any evolution of $u(t,x) \in \mathbb{R}^n$ which is invariant under the affine action of $G \ltimes \mathbb{R}^n$. This evolution induces an evolution on its differential invariants. If $G \neq \mathrm{GL}(n, \mathbb{R})$, and if the evolution induced on the invariants is Hamiltonian with respect to the reduced geometric bracket, then (1.1) is arc-length preserving. Secondly, we will prove that, also for $G \neq \mathrm{GL}(n, \mathbb{R})$, the geometric Poisson brackets are trivial extensions of Poisson brackets for the unparametrized case. That is, it vanishes on functionals that depend only on the parameter of arc-length type and can be restricted to the submanifold where the invariant of arc-length type is constant.

2. Geometric Poisson bracket in Affine geometries

In this section we will briefly describe the definition of geometric Poisson brackets for $M = (G \ltimes \mathbb{R}^n)/G$, G semisimple. More details can be found in [15].

Definition 1. Let $J^k(\mathbb{R}, M)$ the space of k-jets of curves, that is, the set of equivalence classes of curves in M up to k^{th} order of contact. If we denote by $u(x)$ a curve in M and by u_r the r derivative of u with respect to the parameter x, $u_r = \frac{d^r u}{dx^r}$, the jet space has local coordinates that can be represented by $u^{(k)} = (x, u, u_1, u_2, \dots, u_k)$. The group G acts naturally on parametrized curves, therefore it acts naturally on the jet space via the formula

$$g \cdot u^{(k)} = (x, g \cdot u, (g \cdot u)_1, (g \cdot u)_2, \dots)$$

where by $(g \cdot u)_k$ we mean the formula obtained when one differentiates $g \cdot u$ and then writes the result in terms of g, u, u_1, etc. This is usually called the *prolonged* action of G on $J^k(\mathbb{R}, M)$.

Definition 2. A function

$$I : J^k(\mathbb{R}, M) \to \mathbb{R}$$

is called a kth order *differential invariant* if it is invariant with respect to the prolonged action of G.

Definition 3. ([7, 8]) A map

$$\rho : J^k(\mathbb{R}, M) \to G$$

is called a left (resp. right) *moving frame* if it is equivariant with respect to the prolonged action of G on $J^k(\mathbb{R}, M)$ and the left (resp. right) action of G on itself.

If a group acts (locally) *effectively on subsets*, then for k large enough the prolonged action is locally free on regular jets. This guarantees the existence of a moving frame on a neighborhood of a regular jet (for example, on a neighborhood of a generic curve, see [7, 8]).

The group-based moving frame already appears in a familiar method for calculating the curvature of a curve $u(s)$ in the Euclidean plane. In this method one uses a translation to take $u(s)$ to the origin, and a rotation to make one of the axes tangent to the curve. The curvature can classically be found as the coefficient of the second order term in the expansion of the curve around $u(s)$. The crucial observation made in [7, 8] is that *the element of the group* carrying out the translation and rotation depends on u and its derivatives and so it defines a map from the jet space to the group. *This map is a right moving frame*, and it carries all the geometric information of the curve. In fact, Fels and Olver developed a similar normalization process to find right moving frames (see [7, 8] and our next Theorem).

Theorem 1. *(*[7, 8]*) Let $\cdot$ denote the prolonged action of the group on $u^{(k)}$ and assume we have normalization equations of the form*

$$g \cdot u^{(k)} = c_k$$

where c_k are constants (they are called normalization constants*). Assume we have enough normalization equations so as to determine g as a function of $u, u_1, \dots, u_m$. Then $g = g(u^{(m)})$ is a* right invariant moving frame of order m.

This method works equally well if instead of constants we choose c_k depending on differential invariants generators of order less than k. Although we were not able to find this fact explicitly in the literature, it is a consequence of the method in [7, 8]: the transverse section which forms the basis for the normalization method can be equally chosen to be defined by appropriate invariants instead of constants. That is, c_k could as well be differential invariants generated by the differential invariant generators of order less than k (and their derivatives), and the method will be as valid. In particular the generating property (given below) of the resulting moving frame is unchanged.

Definition 4. Consider Kdx to be the horizontal component of the pullback of the left (resp. right)-invariant Maurer-Cartan form of the group G via a group-based left (resp. right) moving frame ρ. That is

$$K = \rho^{-1}\rho_x \in \mathfrak{g} \qquad (\text{resp.} \quad K = \rho_x \rho^{-1})$$

(K is the coefficient matrix of the first order differential equation satisfied by ρ). We call K the *left (resp. right) Serret-Frenet equations* for the moving frame ρ.

Notice that, if ρ is a left moving frame, then ρ^{-1} is a right moving frame and their Serret-Frenet equations are the negative of each other. A complete set of generating differential invariants can always be found among the coefficients of group-based Serret-Frenet equations ([10]). Again, this is also true for moving frames obtained as above with appropriate invariants c_k, rather than merely constants.

The relation between classical moving frames and group-based moving frames is stated in the following theorem.

Theorem 2. *([15]) Let $\Phi_g : G/H \to G/H$ be defined by multiplication by g. That is $\Phi_g([x]) = [gx]$. Let ρ be a group-based left moving frame with $\rho \cdot o = u$ where $o = [H] \in G/H$. Identify $d\Phi_\rho(o)$ with an element of $GL(n)$, where n is the dimension of M.*

Then, the matrix $d\Phi_\rho(o)$ contains in its columns a classical moving frame.

Let us assume now that $M = (G \ltimes \mathbb{R}^n)/G$ so that a left moving frame can be represented as

$$\rho(u^{(k)}) = \begin{pmatrix} 1 & 0 \\ \rho_u & \rho_G \end{pmatrix}$$

where $\rho_u(u^{(k)}) \in \mathbb{R}^n$ and $\rho_G(u^{(k)}) \in G$. If $\rho \cdot 0 = u$, then $\rho_u = u$. With this choice of representation, the Serret Frenet equations are described as

$$K = \rho^{-1}\rho_x = \begin{pmatrix} 0 & 0 \\ \Lambda & K_G \end{pmatrix}$$

The author of [15] showed that Λ contains first order differential invariants. Assume that the first order differential invariants of the curve are all constant, either because the prolonged group action is transitive on an open subset of J^1 (so there are no first order differential invariants, such is the case for $G = \mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{R})$ and $\mathrm{Sp}(n)$) or because we can make them constant by choosing x to be a special arc–length parameter. Such is the case for all other classical groups, since they are the symmetry group of bilinear forms whose associated norm does not vanish generically. In this case $\langle u_1, u_1 \rangle$ is a first order invariant *of arc-length type*, see definition below. Under those assumptions $\rho_G^{-1}(\rho_u)_x = \Lambda \in \mathbb{R}^n$ is constant. (The condition $\Lambda =$ constant is needed to define the geometric Poisson bracket as we will see below.) Define $\mathcal{L}N$ to be the group of Loops on the subspace N, where $N \subset G$ is the isotropy subgroup of Λ, that is

$$N = \{g \in G \text{ such that } g\Lambda = \Lambda\}. \tag{2.1}$$

Let $\mathcal{K}$ be the set of elements of the form $K_G = \rho_G^{-1}(\rho_G)_x$, where ρ_G is a left moving frame associated to curves locally in a neighborhood of u, and where the same normalization constants have been used to find the moving frame for any curve in the neighborhood. That is, K_G contains in its entries a complete set of generating differential invariants. If a special parameter has been fixed, then we assume that all the curves are parametrized by that special choice.

Theorem 3. *([15]) Assume that $\mathcal{L}N$ acts on $\mathcal{L}\mathfrak{g}^*$ with the action*

$$a^*(m)(L) = m^{-1}m_x + m^{-1}Lm.$$

Then, there exists an open subset of $\mathcal{L}\mathfrak{g}^*$, U, such that $U/\mathcal{L}N \cong \mathcal{K}$

The following is a well-known Poisson bracket on functionals defined on $\mathcal{L}\mathfrak{g}^*$. Let $\mathcal{H}, \mathcal{F} : \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ be two functionals. The variational derivative of these functionals can be naturally identified with an element of $\mathcal{L}\mathfrak{g}$ (see [15]), $\frac{\delta\mathcal{H}}{\delta L}(L), \frac{\delta\mathcal{F}}{\delta L}(L) \in \mathcal{L}\mathfrak{g}$. Define the Poisson bracket of $\mathcal{H}$ and $\mathcal{F}$ to be given by

$$\{\mathcal{H},\mathcal{F}\}(L) = \int_{S^1} \left\langle \left(\frac{\delta\mathcal{H}}{\delta L}(L)\right)_x + \mathrm{ad}^*\left(\frac{\delta\mathcal{H}}{\delta L}(L)\right)(L), \frac{\delta\mathcal{F}}{\delta L}(L)\right\rangle dx \tag{2.2}$$

where $\langle,\rangle$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ (for example, the trace if we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$).

Remark 1. Notice that if the Lie algebra $\mathfrak{g}$ is not semisimple then this bracket will be defined only on its semisimple component. Therefore, we always assume that $\mathfrak{g}$ is semisimple. Since $\mathfrak{g} \oplus \mathbb{R}^n$ is not semisimple, one is forced to reduce to the $\mathfrak{g}$ term. This is the reason why the component Λ needs to be constant, so that we can define our Poisson bracket on the space $\mathcal{K}$ generated by the $\mathfrak{g}^*$ component of the Serret-Frenet equations. Also, from now on we will identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using semisimplicity.

The following Theorem was also proved in [15].

Theorem 4. *The bracket (2.2) can be reduced to $\mathcal{K} \cong U/\mathcal{L}N$ to produce a Poisson bracket defined on the space of differential invariants of curves on a neighborhood of a generic curve.*

The practical calculation of this bracket is not too complicated. If the normalization sections are chosen so that the Serret-Frenet equations are simple, calculating explicitly the Poisson bracket can be done algebraically, and in lower dimensions by hand. Indeed, assume $h, f : \mathcal{K} \to \mathbb{R}$ are two functionals defined on $\mathcal{K}$, that is, defined on a submanifold generated by a basis of differential invariants $\mathbf{k} = (k_i)$. Define $h_i = \frac{\delta h}{\delta k_i}$. In order to define the reduction of (2.2) one needs to extend h and f to $\mathcal{H}$ and $\mathcal{F}$ defined on $\mathcal{L}\mathfrak{g}^*$ and constant on the leaves of $\mathcal{L}N$. But one does not need to know explicitly these extensions. Being constant on the leaves of $\mathcal{L}N$ means that

$$\left(\frac{\delta\mathcal{H}}{\delta L}\right)_x + [L, \frac{\delta\mathcal{H}}{\delta L}] \in \mathfrak{n}^0 \tag{2.3}$$

likewise with $\mathcal{F}$, where $\mathfrak{n}$ is the Lie algebra of N and $\mathfrak{n}^0 \subset \mathcal{L}\mathfrak{g}^*$ is its annihilator. When we restrict to $\mathcal{K}$, condition (2.3) determines enough components ofi $\frac{\delta\mathcal{H}}{\delta L}(K_G)$ in terms of h_i and k_i to completely describe the reduced bracket as

$$\{h, f\}_R(\mathbf{k}) = \int_{S^1} \left\langle \left(\frac{\delta\mathcal{H}}{\delta L}(K_G)\right)_x + \left[K_G, \frac{\delta\mathcal{H}}{\delta L}(K_G)\right], \frac{\delta\mathcal{F}}{\delta L}(K_G)\right\rangle dx.$$

We will call the reduced bracket $\{,\}_R$ *a geometric Poisson bracket*. The result in [15] proves that this expression indeed defines a Poisson bracket. The bracket (2.2) has a family of compatible brackets $\{,\}_0$, i.e., brackets such that $\alpha\{,\} + \beta\{,\}_0$ are also Poisson, for any $\alpha, \beta \in \mathbb{R}$. They are given by

$$\{\mathcal{H},\mathcal{F}\}_0 = \int_{S^1} \left\langle \mathrm{ad}^*\left(\frac{\delta\mathcal{H}}{\delta L}(L)\right)(L_0), \frac{\delta\mathcal{F}}{\delta L}(L)\right\rangle dx, \tag{2.4}$$

where $L_0 \in \mathfrak{g}^*$ is any constant element. These companion brackets do not always reduce to $\mathcal{K}$ (the result in [15] stating the opposite is incorrect), and its reduction usually signals the existence of completely integrable evolutions of curves in M.

We give next the theorems in [15] that we will use in this note. Assume we choose moving frames along curves on a neighborhood of a generic curve using the same normalization sections. Assume that

$$\rho_G = (T_1, T_2, \dots T_n)$$

where T_i are the columns of the matrix. Theorem 2 above shows that T_i form an invariant set of independent vectors along the curves. Therefore, it is known ([23]) that any invariant evolution of curves in M can be written as

$$u_t = \rho_G \mathbf{r} = r_1 T_1 + r_2 T_2 + \cdots + r_n T_n \tag{2.5}$$

where $\mathbf{r} = (r_i)$ is a differential invariant vector, i.e., each r_i is generated by $k_1, \dots, k_n$ and their derivatives.

Theorem 5. *([15]) Assume that $u(t,x)$ is a flow solution of the equation (2.5), and assume that there exists $h : \mathcal{K} \to \mathbb{R}$ such that for any extension $\mathcal{H}$ holding (2.3) one has*

$$\frac{\delta \mathcal{H}}{\delta L}(K_G)\Lambda = \mathbf{r}_x + K_G \mathbf{r}. \tag{2.6}$$

Then, the evolution induced on $\mathbf{k}$ by (2.5) is Hamiltonian with respect to the geometric bracket, with Hamiltonian functional h.

3. Hamiltonian structures and invariants of arc-length type

Definition 5. We say a differential invariant k is an invariant of arc-length type if k is a lowest order density, i.e., if $y = \phi(x)$ is any change of variable, then

$$\phi^* k = (\phi_x^s k) \circ \phi^{-1}$$

form some integer s, and s is the lowest nonzero such value.

If we consider $G \subset \mathrm{GL}(n)$ to be a classical group, and if G is the symmetry group of a bilinear form $\langle , \rangle$ such that $\langle v, v \rangle$ does not vanish generically, then $k = \langle u_1, u_1 \rangle$ is a first order differential invariant of a generic curve u, and also an invariant of arc-length type with $s = 2$. As we saw in the previous section, one needs to have k constant in order to be able to define the reduced Poisson bracket. Hence, for the $\mathbf{k}$ evolution to be Hamiltonian and induced by an invariant evolution of curves in M, we will need the evolution to preserve k also.

The question here is: what happens in the other cases, $G = \mathrm{SL}(n, \mathbb{R})$ (equi-affine geometry), $G = \mathrm{Sp}(n)$ (symplectic geometry) and $G = \mathrm{GL}(n, \mathbb{R})$ (general affine geometry)? We will show that in the first two cases there is a differential invariant of arc-length type, k, and that (2.6) implies the preservation of k by the evolution (2.5). In the last case $G = \mathrm{GL}(n, \mathbb{R})$, there is a choice of invariant k, but (2.6) could hold with no invariant of arc-length type evolving trivially under the flow (2.5).

Furthermore, we will also show that in the first two cases the Poisson bracket is in fact a geometric bracket defined on unparametrized curves and trivially extended to the differential invariant of arc-length type. In the last case the bracket is not such a trivial extension, i.e., it is a true parametrized geometric bracket. Other parametrized geometric brackets appear in projective and conformal geometries ([16], [17]), but this is the first example for which the transformation group is of affine form $G \ltimes \mathbb{R}^n$.

3.1. **The case $G = \mathrm{SL}(n, \mathbb{R})$.** The description of group-based Serret-Frenet equations for equi-affine geometry can be found in [15]. In there, it was shown that a left moving frame could be found so that $\rho \cdot 0 = u$ and $\rho \cdot e_i = u_i$, $i = 1, \dots, n-1$ (the vectors e_i represent the standard basis for $\mathbb{R}^n$), $\rho \cdot e_n = \frac{1}{k} u_n$ and such that its associated Serret-Frenet equations are given by

$$K = \begin{pmatrix} 0 & 0 \\ e_1 & K_G \end{pmatrix}$$

where

$$K_G = \begin{pmatrix} 0 & 0 & \dots & 0 & k_1 \\ 1 & 0 & \dots & 0 & k_2 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 1 & 0 & k_{n-1} \\ 0 & \dots & 0 & k & 0 \end{pmatrix}.$$

The invariants $k_i = \det(u_1, \dots, u_{i-1}, u_{n+1}, u_{i+1}, \dots, u_n)$ and $k = \det(u_1, \dots, u_n)$ form a basis for equi-affine differential invariants of parametrized curves. The component ρ_G defining an invariant basis of vectors is given by $\rho_G = (u_1, u_2, \dots, u_{n-1}, \frac{1}{k} u_n)$.

In this case $\Lambda = e_1$ and so the isotropy subgroup N is given by

$$\begin{pmatrix} 1 & * \\ 0 & \Theta \end{pmatrix}$$

with $\Theta \in \mathrm{SL}(n-1, \mathbb{R})$. Its Lie algebra is defined by matrices in $\mathfrak{sl}(n, \mathbb{R})$ with zero first column, and its annihilator $\mathfrak{n}^0$ is given by matrices of the form

$$\begin{pmatrix} \alpha & * \\ 0 & -\frac{\alpha}{n-1} I \end{pmatrix},$$

where I indicates the identity matrix and $\alpha \in C^\infty(S^1)$.

Assume we have an invariant evolution. From [23] we know that this can be written in the form

$$u_t = \sum_{i=1}^{n-1} r_i u_i + \frac{r_n}{k} u_n = \rho_G \mathbf{r} \tag{3.1}$$

where $r_i, i = 1, \dots, n$ are the differential invariants defining the invariant evolution, i.e., functions of k, k_i and their derivatives. Assume a given evolution of this type induces a Hamiltonian evolution on k, k_i, Hamiltonian with respect to the geometric bracket. This means that there exists a Hamiltonian functional h and an extension to $\mathcal{L}\mathfrak{sl}(n)^*$, $\mathcal{H}$, constant on the leaves of $\mathcal{L}N$ and satisfying

$$\left(\frac{\delta \mathcal{H}}{\delta L}(K_G)\right)_x + \left[K_G, \frac{\delta \mathcal{H}}{\delta L}(K_G)\right] \in \mathfrak{n}^0 \tag{3.2}$$

(which is a consequence of $\mathcal{H}$ being constant on the leaves of $\mathcal{L}N$). Also, coming from (3.1), $\mathcal{H}$ will satisfy

$$\frac{\delta \mathcal{H}}{\delta L}(K_G) e_1 = \mathbf{r}_x + K_G \mathbf{r} \tag{3.3}$$

if we are to ensure that the induced evolution on the invariants is Hamiltonian. For more information see [15]. On the other hand, there is a simple way to describe the equation induced on the invariants k, k_i. Indeed, assume $N = \rho^{-1} \rho_t = \begin{pmatrix} 0 & 0 \\ \mathbf{s} & N_G \end{pmatrix}$ represents the evolution of ρ under the flow (3.1). Given that

$\rho^{-1}\rho_t = \rho^{-1}\begin{pmatrix}0 & 0\\ u_t & (\rho_G)_t\end{pmatrix} = \begin{pmatrix}0 & 0\\ \mathbf{r} & *\end{pmatrix}$, we have that $\mathbf{s} = \mathbf{r}$. The compatibility condition, and the fact that $\frac{d}{dx}$ and $\frac{d}{dt}$ commute imply $K_t = N_x + [K, N]$, that is

$$\begin{pmatrix}0 & 0\\ 0 & (K_G)_t\end{pmatrix} = \begin{pmatrix}0 & 0\\ \mathbf{s}_x & (N_G)_x\end{pmatrix} + \left[\begin{pmatrix}0 & 0\\ e_1 & K_G\end{pmatrix}, \begin{pmatrix}0 & 0\\ \mathbf{s} & N_G\end{pmatrix}\right].$$

From here

$$N_G e_1 = \mathbf{r}_x + K_G \mathbf{r}$$

which is the same condition required for $\frac{\delta\mathcal{H}}{\delta L}(K_G)$ in (2.6). Therefore, if a Hamiltonian evolution is induced by a curve evolution, then $\frac{\delta\mathcal{H}}{\delta L}(K_G)e_1 = N_G e_1$.

Theorem 6. *If $\frac{\delta\mathcal{H}}{\delta L}(K_G)e_1 = N_G e_1$ as above, then evolution (3.1) preserves the parameter k.*

Proof. First of all, let N_G be associated to an evolution (3.1) as above. Assume that evolution (3.1) is also Hamiltonian with respect to a Hamiltonian functional f, that is,

$$\mathbf{k}_t = \mathcal{P}\frac{\delta f}{\delta \mathbf{k}}.$$

In that case we know that $N_G e_1 = \frac{\delta\mathcal{F}}{\delta L}(K_G)e_1$. Assume h is *any* other Hamiltonian functional and let $\mathcal{H}$ be an extension such that (3.2) holds true. In that case, since $(K_G)_t = (N_G)_x + [K_G, N_G]$, we have that

$$\left\langle \frac{\delta\mathcal{H}}{\delta L}(K_G), (N_G)_x + [K_G, N_G]\right\rangle = \sum_{i=0}^{n-1} h_i (k_i)_t = \{f, h\}(\mathbf{k}), \tag{3.4}$$

where $h_i = \frac{\delta h}{\delta k_i}$ and $k_0 = k$. This is true since $\mathcal{H}$ is an extension and hence it has h_i located in the dual position to k_i. We will show that, for *any* Hamiltonian functional $\mathcal{H}$, (3.4) above does not depend on $h_0 = \frac{\delta h}{\delta k}$. This will imply that $(k_0)_t = k_t = 0$ and hence evolution (3.1) preserves arc-length.

To see that (3.4) does not depend on $h_0 = \frac{\delta h}{\delta k}$, notice that

$$\left\langle \frac{\delta\mathcal{H}}{\delta L}(K_G), (N_G)_x + [K_G, N_G]\right\rangle = -\left\langle \left(\frac{\delta\mathcal{H}}{\delta L}(K_G)\right)_x + [K_G, \frac{\delta\mathcal{H}}{\delta L}(K_G)], N_G\right\rangle.$$

Therefore, (3.4) will depend on whatever entries of $\frac{\delta\mathcal{H}}{\delta L}(K_G)$ are involved in $\left(\frac{\delta\mathcal{H}}{\delta L}(K_G)\right)_x + [K_G, \frac{\delta\mathcal{H}}{\delta L}(K_G)]$. But

$$\{f, h\}(\mathbf{k}) = \left\langle \left(\frac{\delta\mathcal{F}}{\delta L}(K_G)\right)_x + [K_G, \frac{\delta\mathcal{F}}{\delta L}(K_G)], \frac{\delta\mathcal{H}}{\delta L}(K_G)\right\rangle$$
$$= -\left\langle \left(\frac{\delta\mathcal{H}}{\delta L}(K_G)\right)_x + [K_G, \frac{\delta\mathcal{H}}{\delta L}(K_G)], \frac{\delta\mathcal{F}}{\delta L}(K_G)\right\rangle$$

and so only the entries of $\frac{\delta\mathcal{H}}{\delta L}(K_G)$ in the direction dual to $\mathfrak{n}^0$ will appear in the bracket. Likewise for $\frac{\delta\mathcal{F}}{\delta L}(K_G)$.

Assume now that $\oplus_{i=-n}^{n}\mathfrak{g}_i$ is the standard gradation of $\mathfrak{g}l(n)$ (i.e. $\mathfrak{g}_i$ are matrices with zeroes outside the i diagonal, and the positive and negative gradation corresponds to upper and lower triangular matrices, respectively). Let

$$\frac{\delta\mathcal{H}}{\delta L}(K_G) = \sum_{i=-n}^{n} H_i$$

be the decomposition in terms of the gradation. Then, the entries involved in $\left(\frac{\delta \mathcal{H}}{\delta L}(K_G)\right)_x + [K_G, \frac{\delta \mathcal{H}}{\delta L}(K_G)]$ will be in the non positive part of the gradation, since that is where the dual to $\mathfrak{n}^0$ lies (in fact, it would be entries in the first column and the main diagonal only).

Finally, we can prove that the non positive part of the gradation of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ does not depend on h_0. Indeed, condition (3.2) implies that the components in the negative gradation of $\left(\frac{\delta \mathcal{H}}{\delta L}(K_G)\right)_x + [K_G, \frac{\delta \mathcal{H}}{\delta L}(K_G)]$ will vanish. Let's denote by Γ the matrix below

$$K_G = \Gamma + K_0 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & k & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{k} \\ 0 & 0 \end{pmatrix}.$$

Condition (3.2) implies that

$$\begin{aligned} (H_{-n})_x + [\Gamma, H_{-n+1}] &= 0 \\ (H_{-n+r})_x + [\Gamma, H_{-n+r+1}] + \sum_{i=1}^{r} k_{n-i}[E_{i,n}, H_{-n+r}] &= 0, \end{aligned}$$

for $r = 1, 2, \dots, n-1$, where $E_{i,j}$ has a 1 in place (i,j) and zero elsewhere. These equations show that H_s for $s \leq 0$ depend on k, k_i and the (n,i) entries of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ for $i = 1, \dots, n-1$. Since h_0 is in the $(n-1,n)$ entry of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ and the (n,i) entry is given by h_i, the theorem follows. ♣

Notice that the proof of this theorem also shows that the geometric Poisson bracket $\{h, f\}$ does not depend on $h_0 = \frac{\delta h}{\delta k}$ and $f_0 = \frac{\delta f}{\delta k}$. Therefore, the following theorem is partially proved.

Theorem 7. *The equi-affine geometric Poisson bracket can be restricted to the Poisson submanifold $k = 1$, where k is the equi-affine arc-length. Furthermore, the reduced Poisson bracket is equivalent to the Adler-Gel'fand-Dikii Poisson bracket or second Hamiltonian structure for generalized KdV equations.*

Proof. Since the bracket is independent of h_0, it can be trivially restricted. The only part of the theorem that has not been proved yet is that the resulting reduced bracket is equivalent to the Adler-Gel'fand-Dikii bracket. If $k = 1$ the authors of [4] proved that the Poisson bracket (2.2) for $\mathfrak{g} = \mathfrak{sl}(n+1)$ could be reduced to the affine subspace of $\mathcal{L}\mathfrak{sl}^*(n)$ defined by matrices of the form

$$\begin{pmatrix} 0 & \bar{k}_{n_1} & \dots & \bar{k}_1 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}. \tag{3.5}$$

This affine subspace of $\mathcal{L}\mathfrak{g}^*$ is directly related to ours, as explained in [3], Drinfel'd and Sokolov's are merely a different choice of generating differential invariants. Therefore, Drinfel'd and Sokolov's Poisson bracket (the AGD bracket) is equivalent to our geometric Poisson bracket when written using our set of generating invariants. ♣

Incidentally, the authors of [4] also proved that (2.4) reduces to (3.5) for the choice $L_0 = E_{1,n}$. In fact, for $n = 2$ there exists a completely integrable invariant evolution of equiaffine curves preserving arc-length and inducing the Sowada-Koterra equation on k_1 (see [24]). It is not known if all higher order ones exist.

Example 1. Consider the case $n = 3$. In this case the matrix K_G is given by

$$K_G = \begin{pmatrix} 0 & 0 & k_1 \\ 1 & 0 & k_2 \\ 0 & k & 0 \end{pmatrix}$$

and an invariant evolution of curves will be given by

$$u_t = r_1 u_1 + r_2 u_2 + \frac{r_3}{k} u_3.$$

The matrix describing the evolution of the moving frame under this evolution is determined by $N_G e_1 = \mathbf{r}_x + K_G \mathbf{r}$ (i.e.. n_{i1} are determined), and the relation $(K_G)_t = (N_G)_x + [K_G, N_G]$; That is

$$\begin{pmatrix} n_{11} & * & * \\ n_{21} & n_{22} & * \\ n_{31} & n_{32} & -n_{11} - n_{22} \end{pmatrix}_x + \left[\begin{pmatrix} k_1 n_{31} - n_{12} & * & * \\ k_2 n_{31} + n_{11} - n_{22} & * & * \\ k n_{21} - n_{32} & k(2n_{22} + n_{11}) & k n_{32} \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 0 & (k_1)_t \\ 0 & 0 & (k_2)_t \\ 0 & k_t & 0 \end{pmatrix}. \tag{3.6}$$

The arc-length preserving condition is given by $k_t = 0$ or

$$(n_{32})_x + k(2n_{22} + n_{11}) = 0. \tag{3.7}$$

Using the entries $(2, 1)$ and $(3, 1)$ of (3.6) we obtain that $n_{32} = (n_{31})_x + k n_{21}$ and $n_{22} = (n_{21})_x + k_2 n_{31} + n_{11}$. Therefore, in terms of the first column entries. the arc-length preserving condition is given by

$$(n_{31})_{xx} + (k n_{21})_x + 2k(n_{21})_x + 2k k_2 n_{31} + 3k n_{11} = 0 \tag{3.8}$$

If $h : \mathcal{K} \to \mathbb{R}$ is a Hamiltonian functional and $\mathcal{H}$ is any extension constant on the leaves of $\mathcal{L}N$, then $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ needs to hold the condition (3.2). If $\frac{\delta \mathcal{H}}{\delta L}(K_G) = (h_{ij})$, then this condition becomes

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & -h_{11} - h_{22} \end{pmatrix}_x$$

$$+ \begin{pmatrix} k_1 h_{31} - h_{12} & k_1 h_{32} - k h_{13} & -k_1(2h_{11} + h_{22}) - k_2 h_{12} \\ k_2 h_{31} + h_{11} - h_{22} & k_2 h_{32} + h_{12} - k h_{23} & h_{13} - k_2(h_{11} + 2h_{22}) - k_1 h_{21} \\ k h_{21} - h_{32} & k(2h_{22} + h_{11}) & * \end{pmatrix} \tag{3.9}$$

$$= \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

Being an extension of h implies $h_{23} = \frac{\delta h}{\delta k}$, $h_{31} = \frac{\delta h}{\delta k_1}$ and $h_{32} = \frac{\delta h}{\delta k_2}$ since they are located in the dual positions to k, k_1 and k_2 (we are identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ using the trace and $K \in \mathcal{L}\mathfrak{g}^*$, $\frac{\delta\mathcal{H}}{\delta L}(K_G) \in \mathcal{L}\mathfrak{g}$). Using the entries $(2,1)$, $(3,1)$ and $(3,2)$ of (3.9) we get

$$3kh_{11} = -(h_{32})_x - 2kk_2h_{31} - 2k(h_{21})_x, \quad kh_{21} = h_{32} - (h_{31})_x,$$

and, from here $h_{32} = kh_{21} + (h_{31})_x$ and

$$3kh_{11} = -(kh_{21})_x - (h_{31})_{xx} - 2kk_2h_{31} - 2k(h_{21})_x. \tag{3.10}$$

Imposing a condition of the form $\frac{\delta\mathcal{H}}{\delta L}(K_G)e_1 = \mathbf{r}_x + K_G\mathbf{r} = N_Ge_1$ implies $h_{i1} = n_{i1}$ and, therefore, conditions (3.8) and (3.10) are clearly identical. Hence, for a curve evolution to induce a Hamiltonian evolution on its invariants, the evolution needs to be arc-length preserving.

Notice that using the equations in place $(2,1),(3,1),(3,2)$ and $(2,3)$ we are also able to solve for all entries as functions of h_{23}, h_{31} and h_{32}, i.e., in terms of the variational derivative of h. We obtain

$$\begin{aligned} 3kh_{11} &= -h_{32}' - 2kk_2h_{31} - 2k\left(\frac{1}{k}(h_{32} - h_{31}')\right)' \\ 3kh_{22} &= -h_{32}' + kk_2h_{31} + k\left(\frac{1}{k}(h_{32} - h_{31}')\right)' \\ kh_{13} &= -k_2h_{32}' + k_1(h_{32} - h_{31}') - h_{23}' \\ kh_{21} &= h_{32} - h_{31}'. \end{aligned}$$

Finally, assume that h, f are two functionals and $\mathcal{H}$, $\mathcal{F}$ are two suitable extensions. We will use the same notation for $\mathcal{F}$ as we did for $\mathcal{H}$. We can find the geometric Poisson bracket directly from the values we just found. The geometric bracket is given by

$$\{h,f\}(\mathbf{k}) = \int_{S^1}\left\langle\left(\frac{\delta\mathcal{H}}{\delta L}(K_G)\right)_x + [K_G, \frac{\delta\mathcal{H}}{\delta L}(K_G)], \frac{\delta\mathcal{F}}{\delta L}(K_G)\right\rangle dx = \int_{S^1}\frac{\delta f}{\delta\mathbf{k}}^T\mathcal{P}\frac{\delta h}{\delta\mathbf{k}}dx$$

where $\frac{\delta f}{\delta\mathbf{k}} = (\frac{\delta f}{\delta k}, \frac{\delta f}{\delta k_1}, \frac{\delta f}{\delta k_2})^T$. Substituting all entries and after rather long, but straightforward, calculations one gets that the matrix of differential operators $\mathcal{P}$ defining the bracket is given by

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & P_{11} & P_{12} \\ 0 & -P_{12}^* & P_{22} \end{pmatrix}$$

where

$$\begin{aligned} P_{11} &= \frac{2}{3}D\frac{1}{k}D^3\frac{1}{k}D + \frac{2}{3}(k_2Dk_2 - k_2D^2\frac{1}{k}D - D\frac{1}{k}D^2k_2) + D\frac{1}{k}Dk_1 - k_1D\frac{1}{k}D, \\ P_{12} &= \frac{k_1}{k}D + D\frac{k_1}{k} + k_1D\frac{1}{k} - \frac{1}{3}(D\frac{1}{k}D^2\frac{1}{k}D + 2D\frac{1}{k}D^3\frac{1}{k}) + \frac{1}{3}(k_2D\frac{1}{k}D + 2k_2D^2\frac{1}{k}), \\ P_{22} &= \frac{k_2}{k}D + D\frac{k_2}{k} - \frac{2}{3}(D\frac{1}{k}D\frac{1}{k}D + \frac{1}{k}D^3\frac{1}{k}) - \frac{1}{3}(D\frac{1}{k}D^2\frac{1}{k} + \frac{1}{k}D^2\frac{1}{k}D) \end{aligned}$$

and where $D = \frac{d}{dx}$. As expected, the first row and column vanish and hence we can further restrict the bracket to $k = 1$. When $k = 1$, the restricted bracket is defined by the matrix of differential operators

$$\begin{pmatrix} \frac{2}{3}D^5 + \frac{2}{3}(k_2Dk_2 - k_2D^3 - D^3k_2) + D^2k_1 - k_1D^2 & -D^4 + k_2D^2 + 2k_1D + Dk_1 \\ D^4 - D^2k_2 + 2Dk_1 + k_1D & -2D^3 + k_2D + Dk_2 \end{pmatrix}$$

a bracket which is equivalent to the Adler-Gelfand-Dikii bracket or second Hamiltonian structure for generalized KdV equations. For a description of this equivalency, see [3]. Finally, the restriction of (2.4) with $L_0 = E_{1,3}$ is given by

$$\{h, f\}_0(\mathbf{k}) = \int_{S^1} \left\langle \left[E_{1,3}.\frac{\delta\mathcal{H}}{\delta L}(K_G)\right], \frac{\delta\mathcal{F}}{\delta L}(K_G)\right\rangle dx = \int_{S^1} \frac{\delta f}{\delta \mathbf{k}}^T \mathcal{P}_0 \frac{\delta h}{\delta \mathbf{k}} dx$$

where the operator $\mathcal{P}_0$ is given by

$$\mathcal{P}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{5}{2}\frac{1}{k}D + D\frac{1}{k} \\ 0 & \frac{5}{2}D\frac{1}{k} + \frac{1}{k}D & 0 \end{pmatrix}.$$

This bracket restricts to $k = 1$ to obtain the bracket defined by the differential operator

$$\frac{7}{2}\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$$

which is the well-known companion to the AGD bracket.

3.2. **The case $G = \mathrm{Sp}(2n)$.** As in the previous subsection, we need to determine what is the condition to have a preservation of the arc-length invariant by an evolution of curves $u(t, x) \in \mathbb{R}^n$, invariant under the symplectic group. We then need to show that, if the induced evolution on $\mathbf{k}$ is Hamiltonian, then the condition to preserve the parameter holds true.

In this case a generating system of independent differential invariants is given by

$$\hat{k}_i = u_{i+1}^T J u_{i+2} \tag{3.11}$$

for $i = 0, 1, \ldots, n-1$. Clearly $k = \hat{k}_0$ is an invariant of arc-length type.

Lemma 1. *There exists a left moving frame along a generic curve such that its associated Serret-Frenet equations are defined by the matrix*

$$K = \begin{pmatrix} 0 & 0 \\ e_1 & K_G \end{pmatrix}$$

where

$$K_G = (ke_{n+1}, \ldots, k_1 e_1 + \frac{1}{k}e_2, \ldots)$$

and where the entries not shown are not relevant to the present calculation, the two nonzero columns shown are in place 1 *and* $n+1$*, and the invariant* k_1 *is defined as* $k_1 = -\frac{\hat{k}_1}{k^2}$.

Proof. This lemma is a consequence of Lemma 2 below. ♣

As before, if

$$u_t = \rho_G \mathbf{r} = r_1 T_1 + \cdots + r_n T_n \tag{3.12}$$

is any invariant evolution, then the induced evolution on the differential invariants is given by the equation

$$K_t = N_x + [K, N] \tag{3.13}$$

where $K = \rho^{-1}\rho_x$ and $N = \rho^{-1}\rho_t$. Again, from [15] we know that

$$K = \begin{pmatrix} 0 & 0 \\ e_1 & K_G \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & 0 \\ \mathbf{r} & N_G \end{pmatrix},$$

and condition (3.13) splits into

$$N_G e_1 = \mathbf{r}_x + K_G \mathbf{r} \quad \text{and} \quad (K_G)_t = (N_G)_x + [K_G, N_G].$$

Recall now that the algebra $\mathfrak{sp}(n)$ is represented by matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

where both B and C are symmetric. This means that, if the first column of $K_G \in \mathfrak{sp}(n)$ is ke_{n+1}, then the $n+1$ row of K_G is given by ke_1^T. It also implies that, if $R = (r_{i,j}) \in \mathfrak{sp}(n)$ then $r_{1,1} = -r_{n+1,n+1}$.

Since the invariant k appears in the $(n+1,1)$ entry of K_G, we have that, if $N_G = (n_{i,j})$, the arc-length preserving condition is given by

$$k_t = (n_{n+1,1})_x + kn_{1,1} - kn_{n+1,n+1} = (n_{n+1,1})_x + 2kn_{1,1} = 0 \tag{3.14}$$

or $n_{1,1} = -\frac{1}{2}(n_{n+1,1})_x$. Finally, given a Hamiltonian functional $h : \mathcal{K} \to \mathbb{R}$, the variational derivative of an extension $\mathcal{H}$ constant on the leaves of $\mathcal{L}N$ needs to satisfy

$$\left(\frac{\delta \mathcal{H}}{\delta L}(K_G)\right)_x + \left[K_G, \frac{\delta \mathcal{H}}{\delta L}(K_G)\right] \in \mathfrak{n}^0. \tag{3.15}$$

In this case $\mathfrak{n}$ is the subalgebra of $\mathfrak{sp}(n)$ with zero first column (and hence zero $n+1$ row). Therefore, $\mathfrak{n}^0$ is defined by matrices in $\mathfrak{sp}(n)$ whose only nonzero entries are those of the first row and $n+1$ column. Furthermore, since $\mathcal{H}$ is an extension of h, along $\mathcal{K}$ the $(1, n+1)$ entry of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ is given by the variational derivative of h with respect to k (since k is in the $(n+1,1)$ entry of K_G) and the $(n+1,1)$ entry of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ is given by $h_1 = \frac{\delta h}{\delta k_1}(\mathbf{k})$ (since k_1 is the entry in place $(1, n+1)$ of K_G).

With this information we can now finish the proof of our result. Condition (2.6) is necessary and sufficient for an evolution to induce a Hamiltonian evolution on its differential invariants. Therefore, the condition can be read as

$$\frac{\delta \mathcal{H}}{\delta L}(K_G)e_1 = N_G e_1 = \mathbf{r}_x + K_G \mathbf{r}.$$

The arc-length preserving condition of an evolution is given by (3.14), which becomes $(H_{n+1,1})_x + 2kH_{11} = 0$, where $\frac{\delta \mathcal{H}}{\delta L}(K_G) = (H_{i,j})$. But, using the vanishing of the $(n+1,1)$ entry of the expression (3.15) we have that

$$(H_{n+1,1})_x + kH_{1,1} - kH_{n+1,n+1} = 0$$

which is equivalent to $(H_{n+1,1})_x + 2kH_{11} = 0$, since $\frac{\delta \mathcal{H}}{\delta L}(K_G) \in \mathfrak{sp}(n)$ and so $H_{n+1,n+1} = -H_{1,1}$. We just proved the following theorem.

Theorem 8. *Let $u(t,x)$ be a flow solution of an invariant evolution of the form (3.12), and assume that the differential invariants of the flow satisfy an evolution which is Hamiltonian with respect to the geometric Poisson bracket. Then the evolution is arc-length preserving.*

As before, the fundamental reason why evolutions need to be arc-length preserving is the nature of the geometric Poisson bracket. The S$p(2n)$ geometric Poisson bracket is in fact defined on the submanifold where k constant and it was extended trivially to the entire manifold of differential invariants. That is what our next theorem shows. First, a previous, somehow technical, lemma. Notice that finding Serret-Frenet equations might not be enough to make a Poisson study, quite often we need to find moving frames that result in Serret-Frenet equations as simple as possible. This time constant choices of normalizing cross sections is not sufficient to define simple Serret-Frenet equations. Hence, we need to modify the normalization method used in [7, 8].

Lemma 2. *There exist normalization equations and a (left) moving frame ρ such that the space $\mathcal{K}$ of Serret-Frenet matrices associated to them is the affine subspace of $\mathcal{L}\mathfrak{g}^*$ defined by matrices of the form $K = \begin{pmatrix} 0 & 0 \\ e_1 & K_G \end{pmatrix}$ where*

$$K_G = \begin{pmatrix} 0 & K_1 \\ K_{-1} & 0 \end{pmatrix} \tag{3.16}$$

and where

$$K_1 = \begin{pmatrix} k_1 & 1 & 0 & \dots & 0 \\ 1 & k_3 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & k_{2n-3} & 1 \\ 0 & \dots & 0 & 1 & k_{2n-1} \end{pmatrix}, \quad K_{-1} = \begin{pmatrix} k_0 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & k_{2n-2} \end{pmatrix}.$$

The invariants $\{k_i\}$ form a system of functionally independent and generating differential invariants for curves.

Proof. To prove this lemma we will use a modified normalization process, similar to the one that appears in [7, 8]. In [7, 8] the authors proved that, if enough normalization equations of the form $g \cdot u^{(r)} = c_r$, with c_r constant, can be found to completely determine $\rho(u^{(s)}) = g$, then ρ is a right moving frame for u and the entries of $\rho_x \rho^{-1}$ generate all other differential invariants for the curve. On the other hand, the condition c_r constant is not essential. Indeed, one can use, instead of constants c_r, differential invariants of order less than r, and their derivatives, and the method will still work as well. It will create a right moving frame whose Serret-Frenet equations produce a system of generators for all other differential invariants. As we said in the introductory section, we could not find this published anywhere, but it can be concluded directly from the process in [7, 8]. Here we will use normalization equations for which c_r are differential invariants of lower order. Although not standard, the choices simplify the appearance of K.

The normalization equations and the moving frame will be created following a recursion process. Let a moving frame be given by (we use ρ^{-1} since we are looking for a left moving frame and the process will produce a right one -its inverse-)

$$\rho^{-1} = \begin{pmatrix} 1 & 0 \\ \rho_u & \rho_G \end{pmatrix}$$

The first normalization equation is of course $\rho^{-1} \cdot u = \rho_G \cdot u + \rho_u = 0$ which determines $\rho_u = -\rho_G u$. The rest of the normalization equations are of the form $\rho_G u_i = c_i$. The first two in that group are given by

$$\rho_G u_1 = e_1, \quad \rho_G u_2 = \hat{k}_0 e_{n+1}.$$

the coefficient $\hat{k}_0$ of the second equation is given as in (3.11) and it is determined by the condition $\rho_G \in \mathrm{Sp}(n)$ (since $u_1^T J u_2 = (\rho_G u_1)^T J \rho_G u_2$). Our second group of equations is given by

$$\rho_G u_3 = Y_2 e_2 + \alpha_{2,1} e_1 + \alpha_{2,n+1} e_{n+1}$$

$$\rho_G u_4 = \tilde{k}_2 e_{n+2} + X_2 e_2 + \beta_{2,1} e_1 + \beta_{2,n+1} e_{n+1}$$

where $\alpha_{2,1} = -\frac{\hat{k}_1}{\hat{k}_0}$, $\alpha_{2,n+1} = \hat{k}_0'$, $\beta_{2,1} = -\frac{\hat{k}_1'}{\hat{k}_0}$ and $\beta_{2,n+1} = \hat{k}_0'' - \hat{k}_1$ are also completely determined by $\rho_G \in \mathrm{Sp}(n)$, as before. The generator $\tilde{k}_2$ is also completely determined by the relation $\hat{k}_2 = Y_2 \tilde{k}_2 + \alpha_{2,1}\beta_{2,n+1} - \beta_{2,1}\alpha_{2,n+1}$, once Y_2 has been chosen. If Y_2 depends, as it will, on generators of order less than $\hat{k}_2$, then $\{\hat{k}_0, \hat{k}_1, \hat{k}_2\}$ (functionally) generate the same space of differential invariants as $\{\hat{k}_0, \hat{k}_1, \tilde{k}_2\}$.

These normalization equations define the first four columns of ρ_G^{-1} for *any* choice of X_2 and Y_2. The two normalization terms X_2, Y_2 will be chosen to determine a convenient shape of the Serret-Frenet equations.

The recurrence equations in [7, 8] help us relate the entries of K_G and the invariants $c_r = I_r = \rho_G u_r$. Indeed, if K_G is the $\mathfrak{g}$-block of the left moving frame, then

$$K_G I_r = I_{r+1} - I_r'$$

for any r. This formula is a rewriting in our notation of the formula found in [7, 8]. Using the formula we quickly see that

$$K_G I_1 = K_G e_1 = (\hat{k}_0) e_{n+1} - e_1' = \hat{k}_0 e_{n+1}$$

and

$$K_G I_2 = \hat{k}_0 K_G e_{n+1} = Y_2 e_2 + \alpha_{2,1} e_1 + \alpha_{2,n+1} e_{n+1} - \hat{k}_0' e_{n+1} = Y_2 e_2 + \alpha_{2,1} e_1.$$

If we choose $Y_2 = \hat{k}_0$, $k_1 = \frac{\alpha_{2,1}}{\hat{k}_0} = -\frac{\hat{k}_1}{\hat{k}_0^2}$ and $k_0 = \hat{k}_0$, we have that $K_G e_1 = k_0 e_{n+1}$ and $K_G e_{n+1} = k_1 e_1 + e_2$, with k_0 and k_1 functionally generating the same set of differential invariants as $\hat{k}_0$ and $\hat{k}_1$. Notice that Y_2, which appears in I_3, is determined by a choice of $K_G e_2$. Next

$$K_G I_3 = I_4 - I_3'$$

which becomes, after straightforward calculations,

$$Y_2 K_G e_2 = \tilde{k}_2 e_{n+2} - (\alpha_{2,n+1} + Y_2' - X_2) e_2$$

so that the choice $X_2 = 2\hat{k}_0'$ and $k_2 = \frac{\tilde{k}_2}{Y_2}$ will guarantee that $K_G e_2 = k_2 e_{n+2}$. Also $\{k_0, k_1, k_2\}$ functionally generates the same set of differential invariants as $\hat{k}_0, \hat{k}_1, \hat{k}_2$.

One can now readily see the way to proceed. Assume that normalization equations have been determined of the form

$$I_{2k-1} = \rho_G u_{2k-1}$$

$$= Y_k e_k + \tilde{k}_{2k-3} e_{k-1} + \alpha_{2k-1,n+k-1} e_{n-k+1} + \sum_{s=1}^{k-2} (\alpha_{2k-1,s} e_s + \alpha_{2k-1,n+s} e_{n+s})$$

$$I_{2k} = \rho_G u_{2k} = \hat{k}_{2k-2} e_{n+k} + X_k e_k + \sum_{s=1}^{k-1} (\beta_{2k,s} e_s + \beta_{2k,n+s} e_{n+s}),$$

for $k = 1, 2, \ldots, r$ and Y_k, X_k chosen so that $K_G e_k = k_{2k-2} e_{n+k}$ for $k = 1, \ldots, r$ and $K_G e_{n+k} = k_{2k-1} e_k + e_{k+1} + e_{k-1}$ for $k = 1, \ldots, r-1$. Assume also that $\{\hat{k}_0, \ldots, \hat{k}_{2k-2}\}$ generates the same space of differential invariants as $\{\hat{k}_0, \ldots, \tilde{k}_{2r-2}\}$ and as $\{k_0, \ldots, k_{2r-2}\}$. Likewise with all nested subsets, as before. We define

$$\rho_G u_{2r+1} = Y_{r+1} e_{r+1} + \tilde{k}_{2r-1} e_r + \alpha_{2r+1,n+r} e_{n+r} + \sum_{s=1}^{r-1} (\alpha_{2r+1,s} e_s + \alpha_{2r+1,n+s} e_{n+s})$$

$$\rho_G u_{2r+2} = \hat{k}_{2r} e_{n+r+1} + X_{r+1} e_{r+1} + \sum_{s=1}^{r} (\beta_{2r+2,s} e_s + \beta_{2r+2,n+s} e_{n+s})$$

where all coefficients will be determined by $\rho_G \in \mathrm{Sp}(n)$, except for Y_{r+1} and X_{r+1}, and where $\tilde{k}_{2r-1}$ is also a generator. Notice that, since $K_G \in \mathfrak{sp}(n)$ and $K_G e_k = k_{2k-2} e_{n+k}$ for $k = 1, \dots, r$, the vectors $K_G e_{r+1}$ and $K_G e_{n+r}$ will not depend on e_{n+s} for $s = 1, \dots r$. Also, since $K_G e_{n+k} = k_{2k-1} e_k + e_{k+1} + e_{k-1}$ for $k = 1, \dots, r-1$, $K_G e_{r+1}$ will not depend on e_k, $k = 1, \dots, r-1$ and $K_G e_{n+r}$'s only term in e_k, $k = 1, \dots, r-1$ will be e_{r-1}. This fact will greatly simplify our calculations. Using that $K_G I_{2r} = I_{2r+1} - I_{2r}'$, substituting the values we have and ignoring terms we know will vanish, one gets

$$\hat{k}_{2r-2} K_G e_{n+r} = \hat{k}_{2r-2} e_{r-1} + Y_{r+1} e_{r+1} + (\tilde{k}_{2r-1} - \alpha_{2r-2,n+r-1} - X_r') e_r.$$

Choosing $Y_{r+1} = \hat{k}_{2r-2}$ and $k_{2r-1} = \tilde{k}_{2r-1} - \alpha_{2r-2,n+r-1} - X_r'$, we obtain that $K_G e_{n+r} = k_{2r-1} e_r + e_{r-1} + e_{r+1}$. This condition further implies that $K_G e_{r+1}$ will not depend on e_r either. Finally, since $K_G I_{2r+1} = I_{2r+2} - I_{2r+1}'$, substituting the values we have and ignoring again terms we know will disappear, we get

$$Y_{r+1} K_G e_{r+1} = (X_{r+1} - Y_{r+1}' - \alpha_{2r+1,n+r}) e_{r+1} + \hat{k}_{2r} e_{n+r+1},$$

so that, if we choose $X_{r+1} = Y_{r+1}' + \alpha_{2r+1,n+r}$ and $k_{2r} = \frac{\hat{k}_{2r}}{Y_{r+1}}$ we obtain our last step in this induction process.

Notice that the induction guarantees that, once we fix Y_r and X_r, the values of $K_G e_r$ and $K_G e_{n+r-1}$ get fixed. That means that, once we have completely fixed our moving frame ρ, the values of $K_G e_{2n}$ have not been determined yet by this process. But that is not a problem. Since $K_G \in \mathfrak{sp}(n)$, fixing the first $2n-1$ columns of K_G determines K_G completely except for one entry, the one in place $(n, 2n)$. Also, this process has generated the invariants $k_0, \dots, k_{2n-2}$. Since the entries of K_G are known to generate all other differential invariants, and to attain that one needs $2n$ functionally independent invariants, we call k_{2n-1} the entry in place $(n, 2n)$, and we finish that way the proof of the lemma. ♣

Theorem 9. *Assume h is a functional on $\mathcal{K}$ that depends on $k = k_0$ only. That is, $\frac{\delta h}{\delta k_i} = 0$ for all $i = 1, \dots, n-1$. Then, $\{h, f\}(\mathbf{k}) = 0$ for any functional f on $\mathcal{K}$, where $\{,\}$ is the geometric Poisson bracket of $\mathrm{Sp}(2n)$. Therefore, the geometric Poisson bracket restricts to the submanifold of $\mathcal{K}$ given by $k = 1$ to produce the geometric Poisson bracket for unparametrized curves.*

Proof. If f and h are two Hamiltonian functionals, their geometric Poisson bracket is defined as

$$\{h, f\}(\mathbf{k}) = \int_{S^1} \mathrm{trace} \left(\left(\left(\frac{\delta \mathcal{H}}{\delta L}(K_G) \right)_x + \left[K_G, \frac{\delta \mathcal{H}}{\delta L}(K_G) \right] \right) \frac{\delta \mathcal{F}}{\delta L}(K_G) \right) dx$$

where $\mathcal{H}$ and $\mathcal{F}$ are suitable extensions of the Hamiltonian satisfying (2.3). Given that $\mathcal{H}$ holds (2.3), we can conclude that only the first column and $n+1$ row of $\frac{\delta \mathcal{F}}{\delta L}(K_G)$ are involved in the definition of the reduced bracket. By skew-symmetry the same is true with $\frac{\delta \mathcal{H}}{\delta L}(K_G)$. Therefore, it suffices to show that $h_0 = \frac{\delta h}{\delta k_0}$ does not appear in the first column of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$.

To prove this we use (2.3) and the fact that $\mathfrak{n}^0$ is given by vanishing entries other than those in the first row and $n+1$ column. Assume $H = \frac{\delta\mathcal{H}}{\delta L}(K_G)$ is given by

$$H = \begin{pmatrix} H_0 & H_1 \\ H_{-1} & -H_0^T \end{pmatrix}$$

with H_1 and H_{-1} symmetric. The diagonals of H_1 and H_{-1} are determined by $\frac{\delta h}{\delta k_i}$ and k_0 only appears in the $(1, n+1)$ entry, in the H_1 block. Since

$$[K_G, H] = \begin{pmatrix} K_1H_{-1} - H_1K_{-1} & -K_1H_0^T - H_0K_1 \\ H_0^TK_{-1} + K_{-1}H_0 & K_{-1}H_1 - H_{-1}K_1 \end{pmatrix}$$

one has that h_0 appears only in entries $(1,1), (1,n+1), (n+1,n+1)$ of $H_x + [K_G, H]$, together with the other $\frac{\delta h}{\delta k_i}$, $i \neq 0$ and the entries that are still to be determined. Since none of those equations are used ($\mathfrak{n}^0$ is defined by matrices with zeroes outside the first row and $n+1$ column, and hence only the vanishing entries are used in (2.3) to determine the remaining entries of H), to prove the theorem it suffices to show that (2.3) determines all undetermined entries in H that appear in the bracket.

To prove this is a little tricky. Equation (2.3) becomes, in our notation ($*$ are entries that are not relevant)

$$H_0' + K_1H_{-1} - H_1K_{-1} = \begin{pmatrix} * & * & \dots & * \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{3.17}$$

$$H_{-1}' + K_{-1}H_0 + H_0^TK_{-1} = 0 \tag{3.18}$$

$$H_1' - K_1H_0^T - H_0K_1 = \begin{pmatrix} * & * & \dots & * \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ * & 0 & \dots & 0 \end{pmatrix} \tag{3.19}$$

where the equations equal zero are the ones we will use to determine the remaining entries of H. Let's call $H_0 = (a_{ij})$, $H_1 = (b_{ij})$ and $H_{-1} = (c_{ij})$.

(a) Using the diagonal of (3.18), whose entries are given by $(\frac{\delta h}{\delta k_{2r-2}})_x + 2k_{2r-2}a_{rr} = 0$, $r = 1, \dots, n$, we can solve for the diagonal of H_0, and so all block diagonals are uniquely determined. If we now use the diagonal of (3.17), given by $a_{rr}' + k_{2r-1}\frac{\delta h}{\delta k_{2r-2}} + c_{r+1r} + c_{r-1r} - k_{2r-2}\frac{\delta h}{\delta k_{2r-1}} = 0$, $r = 2, \dots, n$, we obtain equations for $c_{r+1r} + c_{r-1r}$, with the last equation solving for c_{n-1n}. By symmetry, this group of equations will determine the *second diagonal* of H_{-1}. Now, using the diagonal of (3.19), given by $(\frac{\delta h}{\delta k_{2r-2}})_x - 2k_{2r-1}a_{rr} + a_{r+1r} + a_{r-1r} = 0$, we obtain an equation for $a_{r+1r} + a_{r-1r}$, $r = 2, \dots, n$, with the last equation solving for a_{nn-1}. Since H_0 is not symmetric, this does not determine the second diagonal of H_0 yet.

(b) We go now to the second diagonal of (3.18). This is given by $c_{rr+1}' + k_{2r-2}a_{rr+1} + k_{2r}a_{r+1r} = 0$, $r = 1, \dots, n-1$. Since a_{nn+1} is determined, this equation determines a_{n-1n}. Combining these with the equations in (a) solving for $a_{r-1r} + a_{r+1r}$, we can now solve for *both second diagonals* of H_0.

We then go directly to both second diagonals of (3.17). The lower one has entries $a_{rr-1}' + k_{2r-1}c_{rr-1} + c_{r-1r-1} + c_{r+1r-1} - k_{2r-3}b_{rr-1} = 0$, $r = 2, 3, \dots n$, determining completely *the second diagonal* of H_1. The upper diagonal has entries

$a'_{rr+1}+k_{2r-1}c_{rr+1}+c_{r+1r+1}-k_{2r-2}b_{rr+1}=0$ which determined the third diagonal of H_{-1}.

(c) Repeating this process with the remaining diagonals we can uniquely determine all entries of H using the vanishing equations en (3.17), (3.18) and (3.19). There is probably a more elegant way to describe this fact using the standard gradation of $\mathfrak{gl}(n)$, but we could not find a simpler one. ♣

Example 2. Assume $n=2$. In this case one can choose normalization constant $c_1=e_1$, $c_2=ke_3$, $c_3=ke_2-\frac{\hat{k}_1}{k}e_1+k'e_3$ and $c_4=\tilde{k}_2e_4+2k'e_2-\frac{\hat{k}_1'}{k}e_1+(k''-\hat{k}_1)e_3$, where $k\tilde{k}_2=\hat{k}_2+\frac{\hat{k}_1}{k}(k''-\hat{k}_1)+\frac{1}{k}\hat{k}_1'k'$. With these choices the G component of the left moving frame is given by

$$\rho_G e_1=u_1, \rho_G(ke_3)=u_2,$$
$$\rho_G(ke_2-\frac{\hat{k}_1}{k}e_1+k'e_3)=u_3, \rho_G(\tilde{k}_2e_4+2k'e_2-\frac{\hat{k}_1'}{k}e_1+(k''-\hat{k}_1)e_3)=u_4.$$

This determines ρ_G to be given by the matrix $\rho_G=(T_1T_2T_3T_4)$ where

$$\begin{gathered}T_1=u_1\\ T_2=\tfrac{1}{k}u_3+\tfrac{\hat{k}_1}{k^2}u_1-\tfrac{k'}{k^2}u_2\\ T_3=\tfrac{1}{k}u_2\\ T_4=\tfrac{1}{\tilde{k}_2}u_4-\tfrac{2k'}{k\tilde{k}_2}u_3+(\tfrac{2(k')^2}{k^2\tilde{k}_2}+\tfrac{\hat{k}_1-k''}{k\hat{k}_2})u_2-(\tfrac{\hat{k}_1'}{k\hat{k}_2}+2\tfrac{k'\hat{k}_1}{k^2\tilde{k}_2})u_1\end{gathered} \quad . \tag{3.20}$$

Its Serret-Frenet equation is given by

$$\rho^{-1}\rho_x=\begin{pmatrix}0&0\\ e_1&K_G\end{pmatrix},\quad K_G=\begin{pmatrix}0&0&k_1&1\\ 0&0&1&k_3\\ k&0&0&0\\ 0&k_2&0&0\end{pmatrix}.$$

A basis of differential invariants is given by k, $k_1=-\frac{\hat{k}_1}{k^2}$, $k_2=\frac{1}{k^2}\left(\tilde{k}_2+\frac{\hat{k}_1}{k}(k''-\hat{k}_1)+\frac{\hat{k}_1'k'}{k}\right)$ and $k_3=-\frac{1}{\tilde{k}_2^2}\hat{k}_3+S$, where S depends on k, $\hat{k}_1$, $\hat{k}_2$ and their derivatives. The term S can be found with straightforward calculations from the data above, but its form is long and irrelevant here. As before $\Lambda=e_1$ and so its isotropy algebra $\mathfrak{n}$ are those matrices in $\mathfrak{sp}(n)$ with vanishing first column. If h is a Hamiltonian functional depending on k,k_1,k_2,k_3 and their derivatives, and if $\mathcal{H}$ is an extension to $\mathcal{L}\mathfrak{sp}(n)^*$ constant on the leaves of $\mathcal{L}N$, necessarily $\frac{\delta\mathcal{H}}{\delta L}(K_G)$ must hold (2.3). If $\frac{\delta h}{\delta k}=\hat{h}$, $\frac{\delta h}{\delta k_i}=h_i$, this implies ($*$ indicates entries that are not relevant)

(3.21)
$$\frac{d}{dx}\begin{pmatrix}a_{11}&a_{12}&h&b\\ a_{21}&a_{22}&b&h_2\\ h_1&a&-a_{11}&-a_{21}\\ a&h_3&-a_{12}&-a_{22}\end{pmatrix}+\left[\begin{pmatrix}0&0&k_1&1\\ 0&0&1&k_3\\ k&0&0&0\\ 0&k_2&0&0\end{pmatrix},\begin{pmatrix}a_{11}&a_{12}&h&b\\ a_{21}&a_{22}&b&h_2\\ h_1&a&-a_{11}&-a_{21}\\ a&h_3&-a_{12}&-a_{22}\end{pmatrix}\right]$$
$$=\begin{pmatrix}*&*&*&*\\ 0&0&*&0\\ 0&0&*&0\\ 0&0&*&0\end{pmatrix}.$$

We can now calculate the $(3,1)$, $(4,2)$, $(2,4)$ and $(2,2)$ entries of this equation to obtain

$$h_1' + 2ka_{11} = 0, \qquad h_3' + 2k_2a_{22} = 0,$$
$$h_2' - 2a_{21} - 2k_3a_{22} = 0 \quad a_{22}' - a - k_3h_3 + k_2h_2 = 0.$$

These equations allow us to solve for a_{11}, a_{21}, a_{22} and a *in terms of* h_1, h_2 *and* h_3 *only.* Therefore, equations (2.6) are over determined in h_i and there will be one compatibility condition that needs to be satisfied. Now, one can find the general form of any invariant evolution of curves in $\mathbb{R}^4$ which are invariant under the action of $\mathrm{Sp}(2) \ltimes \mathbb{R}^4$. Indeed, this is given by (3.20). Any such invariant evolution is of the form

$$u_t = \rho_G \mathbf{r} = r_1T_1 + r_2T_2 + r_3T_3 + r_4T_4 \tag{3.22}$$

where T_i are as in (3.20) and where r_i are differential invariants (and hence functions of k, k_1, k_2, k_3 and their derivatives). If $u(t,x)$ is a flow solution of this equation, one can write (2.6) explicitly to find that the evolution induced on its invariants will be Hamiltonian whenever

$$\begin{pmatrix} -\frac{1}{2k}h_1' \\ \frac{k}{2}h_2' + \frac{kk_3}{2k_2}h_3' + \frac{k_3k'}{k_2}h_3 \\ h_1 \\ \frac{k}{2}\left(\frac{1}{k_2}h_3'\right)' + k\left(\frac{k'}{kk_2}h_3\right)' - kk_3h_3 + kk_2h_2 \end{pmatrix} = \begin{pmatrix} r_1' + k_1r_3 + \frac{1}{k}r_4 \\ r_2' - \frac{k'}{k}r_2 + \frac{1}{k}r_3 + k_3r_4 \\ r_3' + kr_1 \\ r_4' + k_2r_2 + \frac{k'}{k}r_4 \end{pmatrix}.$$

Clearly, the compatibility condition for this to be held is given by the first and third equations. That is, we need the condition

$$r_3'' + (kr_1)' = -2k(r_1' + k_1r_3 + \frac{1}{k}r_4).$$

A somehow long but straightforward calculation shows that this is exactly the condition needed for (3.22) to preserve $k = u_1^TJu_2$.

For completion we will provide the geometric Poisson bracket for this case. Notice that Poisson brackets for this dimension are unusual in the literature, most brackets are one or two dimensional (this will be four, restricting to three). Equations (3.21) produce more than the first column of $\frac{\delta\mathcal{H}}{\delta L}(K_G)$, it produces all relevant entries of $\frac{\delta\mathcal{H}}{\delta L}(K_G)$ needed to define explicitly the geometric Poisson bracket. Notice that we have used only four of the six equations available. If we further use the equations in entries $(2,1)$ and $(4,1)$ we also obtain the expressions for b and a_{12} in terms of h_i. We get

$$\begin{aligned} kb &= \frac{1}{2}h_2'' + \frac{1}{2}(\frac{k_3}{k_2}h_3')' + h_1 + k_3k_2h_2 - k_3^2h_3 + \frac{1}{2}k_3(\frac{1}{k_2}h_3')' \\ ka_{12} &= -(k_2h_2)' + (k_3h_3)' - \frac{1}{2}(\frac{1}{k_2}h_3')'' - \frac{1}{2}k_2h_2' - \frac{1}{2}k_3h_3' \end{aligned}$$

We now have all relevant entries in $H = \frac{\delta\mathcal{H}}{\delta L}(K_G)$. If we assume $F = \frac{\delta\mathcal{F}}{\delta L}(K_G)$ to be the analogous one for a different Hamiltonian functional f, we can substitute in (2.2) to obtain the geometric bracket explicitly. The calculations are very long and tedious, but again straightforward. The result is given by

$$\{h, f\}(\mathbf{k}) = \int_{S^1} \frac{\delta f}{\delta \mathbf{k}}(\mathbf{k})^T \mathcal{P} \frac{\delta h}{\delta \mathbf{k}}(\mathbf{k}) dx$$

where $\mathbf{k} = (k, k_1, k_2, k_3)$, $\frac{\delta f}{\delta \mathbf{k}}(\mathbf{k}) = (\frac{\delta f}{\delta k}(\mathbf{k}), \frac{\delta f}{\delta k_1}(\mathbf{k}), \frac{\delta f}{\delta k_2}(\mathbf{k}), \frac{\delta f}{\delta k_3}(\mathbf{k}))^T$, and

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P_{11} & P_{12} & P_{13} \\ 0 & -P_{21}^* & P_{22} & P_{23} \\ 0 & -P_{13}^* & -P_{23}^* & P_{33} \end{pmatrix}.$$

As it was proved above, the bracket restricts to the submanifold where k is constant. For simplicity we will give the explicit expression of $\mathcal{P}$ for $k = 1$. They are

$$\begin{aligned}
P_{11} &= -\frac{1}{2}D^3 + Dk_1 + k_1 D \\
P_{12} &= k_2 D + 3Dk_2 \\
P_{13} &= k_3 D - 3Dk_3 + \frac{3}{2}D^2\frac{1}{k_2}D \\
P_{22} &= D^3 k_2 + k_2 D^3 + \frac{1}{2}(Dk_2 D^2 + D^2 k_2 D) + 2k_2(k_3 Dk_2 + Dk_3 k_2) \\
&+ Dk_2^2 k_3 + k_3 k_2^2 D - (Dk_1 k_2 + k_1 k_2 D) \\
P_{23} &= \frac{1}{2}D^4\frac{1}{k_2}D - D^3 k_3 + \frac{1}{2}D^2 k_3 D + k_2 D^2 \frac{k_3}{k_2}D + k_2 Dk_3 D\frac{1}{k_2}D + k_3 k_2 D^2 \frac{1}{k_2}D \\
&+ \frac{1}{2}(Dk_2 D\frac{k_3}{k_2}D + Dk_3 k_2 D\frac{1}{k_2}D) + Dk_1 k_3 - k_1 k_3 D - \frac{1}{2}Dk_1 D\frac{1}{k_2}D \\
&+ k_2 k_3^2 D - Dk_3^2 k_2 - 2k_2(Dk_3^2 + k_3 Dk_3) \\
P_{33} &= \frac{1}{2}(D\frac{k_3}{k_2}D^3\frac{1}{k_2}D + D\frac{1}{k_2}D^3\frac{k_3}{k_2}D + D\frac{1}{k_2}D^2 k_3 D\frac{1}{k_2}D + D\frac{1}{k_2}Dk_3 D^2\frac{1}{k_2}D) \\
&- (D\frac{k_3}{k_2}D^2 k_3 + k_3 D^2\frac{k_3}{k_2}D + k_3 Dk_3 D\frac{1}{k_2}D + D\frac{1}{k_2}Dk_3 Dk_3 + k_3^2 D^2\frac{1}{k_2}D + D\frac{1}{k_2}D^2 k_3^2) \\
&+ \frac{1}{2}(D\frac{k_3}{k_2}Dk_3 D + Dk_3 D\frac{k_3}{k_2}D) + \frac{1}{2}(D\frac{1}{k_2}Dk_3^2 D + Dk_3^2 D\frac{1}{k_2}D) + \frac{1}{2}D\frac{1}{k_2}D\frac{1}{k_2}D \\
&- \frac{1}{2}(D\frac{k_3 k_1}{k_2}D\frac{1}{k_2}D + D\frac{1}{k_2}D\frac{k_3 k_1}{k_2}D) + D\frac{k_3^2 k_1}{k_2} + \frac{k_1 k_3^2}{k_2}D - (Dk_3^3 + k_3^3 D) \\
&+ 2(k_3 Dk_3^2 + k_3^2 Dk_3) - (\frac{k_3}{k_2}D + D\frac{k_3}{k_2})
\end{aligned}$$

In this case we also have a companion bracket. If we choose $L_0 = E_{2,4}$, the reduction of (2.4) to $\mathcal{K}$, if indeed a Poisson bracket, would be defined as

$$\{h, f\}_0(\mathbf{k}) = \int_{S^1} \left\langle [E_{2,4}, \frac{\delta \mathcal{H}}{\delta L}(K_G)], \frac{\delta \mathcal{F}}{\delta L}(K_G) \right\rangle dx$$

$$= 2\int_{S^1} (h_1 a_{11}^f + 2a^h a_{21}^f - a_{21}^h a^f - a_{11}^h f_1) dx = 2\int_{S^1} \begin{pmatrix} f_0 & f_1 & f_2 & f_3 \end{pmatrix} \mathcal{P}_0 \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix} dx$$

where $h_i = \frac{\delta h}{\delta k_i}$, $k_0 = k$ and where $\mathcal{P}_0$ is the matrix of differential operators

$$\mathcal{P}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{k}D + D\frac{1}{k} & 0 & 0 \\ 0 & 0 & -Dk_2 - k_2 D & Dk_3 - k_3 D + \frac{1}{2}D^2\frac{1}{k_2}D \\ 0 & 0 & k_3 D - Dk_3 + \frac{1}{2}D\frac{1}{k_2}D^2 & X \end{pmatrix} \tag{3.23}$$

with $X = D\frac{k_3}{k_2}\left(k_3 + \frac{1}{2}D\frac{1}{k_2}D\right) + \left(k_3 + \frac{1}{2}D\frac{1}{k_2}D\right)\frac{k_3}{k_2}D$. This bracket is not guaranteed to be Poisson or to reduce to $k=1$, but in our case it is both Poisson and it clearly reduces to $k=1$. In fact, it is again a trivial extension of an unparametrized bracket. One can check that the bracket is Poisson using the techniques described in [23] and used, for example, in [21]. The calculations are long but straightforward so we will not include them. This bracket provides a compatible companion to the geometric Poisson previously obtained.

3.3. **The case $G = \mathrm{GL}(n, \mathbb{R})$.** This case is in some sense similar to $G = \mathrm{SL}(n, \mathbb{R})$, but, surprisingly, no restriction on the coefficients of the evolution (2.5) is needed in advance to obtain Hamiltonian evolutions. In particular, we do not need to have the preservation of a differential invariant of arc-length type.

As before, we can obtain a right moving frame ρ^{-1} using the normalization conditions

$$\rho^{-1}u = 0, \quad \rho^{-1}u_k = e_k \qquad k = 1, \dots, n$$

so that $\rho = (u_1, \dots, u_n)$. Also $\Lambda = e_1$ since it is always equal to the first normalization constant. In that case, the Serret-Frenet equations are defined by the matrix

$$K = \rho^{-1}\rho_x = \begin{pmatrix} 0 & 0 & \dots & 0 & k_1 \\ 1 & 0 & \dots & 0 & k_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & k_{n-1} \\ 0 & \dots & 0 & 1 & k_n \end{pmatrix}$$

where the generating differential invariants are of the form

$$k_i = \frac{1}{d}\det(u_1, \dots u_{i-1}, u_{n+1}, u_{i+1}, \dots, u_n), \quad d = \det(u_1, \dots, u_n), \quad i = 1, \dots n.$$

Observe that none of these invariants have the property $\phi^* k_r = (\phi_1^s k_r) \circ \phi^{-1}$, a higher order combination needs to be put together to accomplish this. See our example below.

The first column of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ is determined by (2.3), that is, if

$$\frac{\delta \mathcal{H}}{\delta L}(K_G) = \begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_{n-1}^T \\ \mathbf{h}^T \end{pmatrix}$$

where v_i are the rows of the matrix, and $\mathbf{h} = \frac{\delta h}{\delta \mathbf{k}}(\mathbf{k}) = (h_i)$, then

$$\frac{d}{dx}\begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_{n-1}^T \\ \mathbf{h}^T \end{pmatrix} + [\begin{pmatrix} k_1 e_n^T \\ \vdots \\ k_2 e_n^T + e_1^T \\ \vdots \\ k_n e_n^T + e_{n-1}^T \end{pmatrix}, \begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_{n-1}^T \\ \mathbf{h}^T \end{pmatrix}] = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \in \mathfrak{n}^0.$$

This relation completely determines $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ to be

$$\mathbf{v}_{n-1} = (h_2, h_3, \dots, h_n, \mathbf{h} \cdot \mathbf{k})^T - k_n \mathbf{h} - \mathbf{h}_x$$
$$\mathbf{v}_r = (\mathbf{v}_{r+1}^2, \mathbf{v}_{r+1}^3, \dots, \mathbf{v}_{r+1}^n, \mathbf{v}_{r+1} \cdot \mathbf{k})^T - k_{r+1}\mathbf{h} - (\mathbf{v}_{r+1})_x$$

$r = 1, 2, \ldots, n-2$. Here we denote the entries of $\mathbf{v}_r$ by $\mathbf{v}_r = (\mathbf{v}_r^i)$. Therefore, the first column of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ is given by $\frac{\delta \mathcal{H}}{\delta L}(K_G)e_1 = (\mathbf{v}_1^1, \ldots, \mathbf{v}_{n-2}^1, h_2 - h_1' - k_n h_1, h_1)^T$. Again, $h_i = \frac{\delta h}{\delta k_i}$. When one looks closely, $\mathbf{v}_r^1 = h_{n-r+1} + Z_r$, where Z_r depends on h_s, $s < n - r + 1$ and their derivatives. Hence, condition (2.6) can be written as

$$\mathbf{h} = F(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \ldots) \tag{3.24}$$

for a certain choice of F (the precise form of F can be found in each particular case). The following theorem follows.

Theorem 10. *A* $\mathrm{GL}(n, \mathbb{R})$ *invariant evolution is of the form*

$$u_t = \sum_{i=1}^{n} r_i u_i = (u_1 \ldots u_n)\mathbf{r} \tag{3.25}$$

where $\mathbf{r} = (r_i)$ *depends on* k_j *and their derivatives. Given any such evolution, the evolution induced on the differential invariants of the flow is Hamiltonian with respect to the geometric Poisson bracket if, and only if, equation (3.24) produces the variational derivative of a Hamiltonian functional* h*, i.e.,* $\mathbf{h} = \frac{\delta h}{\delta \mathbf{k}} = (h_i)$.

On the other hand, this is the only condition needed to produce a Hamiltonian evolution. Indeed, the following example shows how no differential invariant needs to be preserved by all Hamiltonian evolutions and how, in fact, the Hamiltonian vector fields of functionals that depend only on the lowest order invariant of arc-length type do not vanish in general, even in the simpler case $n = 2$.

Example 3. Let's consider $n = 2$. In that case the Serret-Frenet matrix obtained by choosing normalization constants $c_0 = 0, c_1 = e_1 = \Lambda, c_2 = e_2$ is given by

$$K = \begin{pmatrix} 0 & 0 \\ e_1 & K_G \end{pmatrix}, \quad K_G = \begin{pmatrix} 0 & k_1 \\ 1 & k_2 \end{pmatrix}.$$

The isotropy subgroup of $\Lambda = e_1$ is, in this case, $N = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$. Its Lie algebra is given by matrices with vanishing first column. Hence, $\mathfrak{n}^0$ is generated by matrices with vanishing last row. Let $h(k_1, k_2)$ be a functional defined on the space of differential invariants. Then, if $\mathcal{H}$ is an extension to $\mathcal{L}\mathfrak{gl}(n)^* \cong \mathcal{L}\mathfrak{gl}(n)$ and constant on the leaves of $\mathcal{L}N$, its variational derivative $\frac{\delta h}{\delta L}(K_G) = \begin{pmatrix} a & b \\ h_1 & h_2 \end{pmatrix}$ must satisfy

$$\frac{d}{dx}\begin{pmatrix} a & b \\ h_1 & h_2 \end{pmatrix} + \left[\begin{pmatrix} 0 & k_1 \\ 1 & k_2 \end{pmatrix}, \begin{pmatrix} a & b \\ h_1 & h_2 \end{pmatrix}\right] = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}.$$

This relation determines $a = h_2 - h_1' - k_2 h_1$ and $b = -h_2' + k_1 h_1$. Therefore, the first column of $\frac{\delta \mathcal{H}}{\delta L}(K_G)$ is given by $(h_2 - h_1' - k_2 h_1, h_1)^T$. From here, an evolution of the form $u_t = r_1 u_1 + r_2 u_2$, where r_1 and r_2 are functions of k_1, k_2 and their derivatives, induces a Hamiltonian evolution on k_1, k_2 whenever

$$\begin{pmatrix} h_2 - h_1' - k_2 h_1 \\ h_1 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}_x + \begin{pmatrix} 0 & k_1 \\ 1 & k_2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} r_1' + k_1 r_2 \\ r_2' + r_1 + k_2 r_2 \end{pmatrix}. \tag{3.26}$$

That is, $h_1 = r_2' + r_1 + k_2 r_2$ and $h_2 = r_2'' + (k_2' + k_2^2 + k_1) r_2 + 2k_2 r_2' + k_2 r_1 + 2r_1'$. For those evolutions for which $h_1 = \frac{\delta h}{\delta k_1}$ and $h_2 = \frac{\delta h}{\delta k_2}$, for some Hamiltonian h, the evolution of (k_1, k_2) will be Hamiltonian with respect to the geometric Poisson bracket, with Hamiltonian functional h. A straightforward calculation shows that, if h and f are two Hamiltonian functionals, and $\mathcal{H}$, $\mathcal{F}$ are two extensions that are constant on the leaves of $\mathcal{L}N$, then the geometric Poisson bracket is found as

$$\{h,f\}(k_1,k_2) = \int_{S^1} \operatorname{trace}\left(\left(\left(\frac{\delta\mathcal{H}}{\delta L}(K_G)\right)_x + \left[\begin{pmatrix}0 & k_1\\ 1 & k_2\end{pmatrix}, \frac{\delta\mathcal{H}}{\delta L}(K_G)\right]\right)\frac{\delta\mathcal{F}}{\delta L}(K_G)\right) dx$$

Using the expression for $\frac{\delta\mathcal{H}}{\delta L}(K_G)$ and $\frac{\delta\mathcal{F}}{\delta L}(K_G)$ found above we have that the explicit geometric Hamiltonian structure is given by

$$\{h,f\}(k_1,k_2) = \int_{S^1} \begin{pmatrix}g_1 & g_2\end{pmatrix} \mathcal{P} \begin{pmatrix}h_1\\ h_2\end{pmatrix} dx.$$

where

$$\mathcal{P} = \begin{pmatrix} -D^3 + k_1 D + Dk_1 - D^2 k_2 + k_2 D^2 + k_2 D k_2 & D^2 - k_2 D\\ -D^2 - Dk_2 & 2D\end{pmatrix}$$

is the matrix of differential operators defining the Poisson structure.

Finally, one can directly check that a differential invariant of arc-length type (that is, of lowest order) is given by $k = k_1 - \frac{1}{3}k_2' + \frac{2}{9}k_2^2$. Given that (from (3.26)) an evolution (3.25) induces the Hamiltonian evolution

$$\begin{pmatrix}k_1\\ k_2\end{pmatrix}_t = \mathcal{P}\begin{pmatrix}h_1\\ h_2\end{pmatrix} = \mathcal{P}\begin{pmatrix} r_2' + r_1 + k_2 r_2\\ r_2'' + (k_2' + k_2^2 + k_1) r_2 + 2k_2 r_2' + k_2 r_1 + 2r_1'\end{pmatrix}$$

we can readily see that the evolution of k is given by

$$\begin{aligned} k_t &= \left(-\frac{2}{3}D^3 + kD + Dk + \frac{2}{9}k_2 D^2 - \frac{1}{3}D^2 k_2 + \frac{5}{9}k_2 D k_2 - \frac{2}{9}k_2^2 D - \frac{2}{9}Dk_2^2\right) h_1\\ &+ \left(\frac{1}{3}D^2 - \frac{1}{9}k_2 D\right) h_2 \end{aligned}$$

which does not vanish in general. In particular, and unlike the previous examples, the geometric Poisson bracket is not a trivial extension to the parametrized case of a Poisson bracket on unparametrized curves.

For completion, notice that, if we choose an arbitrary $L_0 = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, the reduction of (2.4) is given by

$$\{h,f\}(k_1,k_2) = \int_{S^1} \left\langle \left[L_0, \frac{\delta\mathcal{H}}{\delta L}(K_G)\right], \frac{\delta\mathcal{F}}{\delta L}(K_G)\right\rangle dx = \int_{S^1} \begin{pmatrix}f_1 & f_2\end{pmatrix} \mathcal{P}_0 \begin{pmatrix}h_1\\ h_2\end{pmatrix} dx$$

where

$$\mathcal{P}_0 = (d-a)\begin{pmatrix}0 & D\\ D & 0\end{pmatrix} + 2b\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix} - c\begin{pmatrix}Dk_1 + k_1 D & -D^2 + k_2 D\\ D^2 + Dk_2 & 0\end{pmatrix}.$$

The matrix $\mathcal{P}_0$ defines a Hamiltonian structure for any values of a, b, c, d.

References

[1] S. Anco. *Hamiltonian flows of curves in G/SO(N) and vector soliton equations of mKdV and sine-Gordon type,* SIGMA, **2** (2006) 004.

[2] S. Anco. *Bi-Hamiltonian operators, integrable flows of curves using moving frames, and geometric map equations,* J. Phys. A: Math. Gen., **39** (2006), 2043-2072.

[3] A. Calini, T. Ivey and G. Marí Beffa. *Remarks on KdV-type flows on star-shaped curves,* in preparation.

[4] V. G. Drinfel'd and V. V. Sokolov. *Lie algebras and equations of Korteweg– de Vries type,* In Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, pages 81–180. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[5] A. Doliwa and P.M. Santini. *An elementary geometric characterization of the integrable motions of a curve*, Phys. Lett. A, **185** 4, (1994) 373–384.
[6] Ferapontov. *Isoparametric hypersurfaces in spheres, integrable non-diagonalizable systems of hydrodynamic type, and N-wave systems*, Diff. Geom. Appl. **5** (1995), 335-369.
[7] M. Fels, P.J. Olver, *Moving coframes. I. A practical algorithm*, Acta Appl. Math. (1997), pp 99–136.
[8] M. Fels, P.J. Olver, *Moving coframes. II. Regularization and theoretical foundations*, Acta Appl. Math. (1999), pp 127–208.
[9] F. Gay-Balmaz, T.S. Ratiu. *Group actions on chains of Banach manifolds and applications to fluid dynamics*, Ann. Global Anal. Geom, to appear.
[10] E. Hubert. *Generation properties of differential invariants in the moving frame methods*, preprint.
[11] Kai-Seng Chou and C. Qu. *Integrable equations arising from motions of plane curves*, Phys. D, **162**(1-2), (2002) 9-33.
[12] Kai-Seng Chou and C. Qu. *Integrable equations arising from motions of plane curves II*, J. Nonlinear Sci., **13**(15), (2003) 487-517.
[13] J. Langer and R. Perline. *Poisson geometry of the filament equation*, J. Nonlinear Sci., **1**(1), (1991) 71–93.
[14] J. Langer and R. Perline. *Geometric realizations of Fordy-Kulish nonlinear Schrödinger systems*, Pacific Journal of Math., **195**(1), (2000) 157-178.
[15] G. Marí Beffa, *Poisson geometry of differential invariants of curves in some nonsemisimple homogenous spaces*, Proc. Amer. Math. Soc. 134 (2006) 779-791.
[16] G. Marí Beffa, *The theory of differential invariants and KdV Hamiltonian evolutions*, Bull. Soc. Math. France, 127(3) (1999) 363–391.
[17] G. Marí Beffa, *Poisson brackets associated to the conformal geometry of curves*, Trans. Amer. Math. Soc. 357 (2005) 2799-2827.
[18] G. Marí Beffa, *Hamiltonian Structures on the space of differential invariants of curves in flat semisimple homogenous manifolds*, accepted for publication in the Asian Journal of Mathematics.
[19] G. Marí Beffa, *Projective-type differential invariants and geometric curve evolutions of KdV-type in flat homogeneous manifolds*, accepted for publication in the Annales de l'Institute Fourier.
[20] G. Marí Beffa, On completely integrable geometric evolutions of curves of Lagrangian planes, *Proceedings of the Royal Society of Edinburg*, **137A** (2007), 111–131.
[21] G. Marí Beffa, J. Sanders, J.P. Wang, Integrable Systems in Three–Dimensional Riemannian Geometry, *J. Nonlinear Sc.*, (2002) pp 143–167.
[22] J.E. Marsden and A. Weinstein. *Coadjoint orbits, vortices and Clebsch variables for incompressible fluids*, Phys. D, **7** (1-3) (1983), 305-323.
[23] P.J. Olver, *Equivalence, Invariance and Symmetry*, Cambridge University Press, Cambridge, UK, 1995.
[24] P.J. Olver, *Invariant submanifold flows*, J. Phys. A, to appear
[25] J. Sanders and J.P. Wang, *Integrable Systems in n-dimensional Riemannian Geometry*, Moscow Mathematical Journal, 3, 2004.
[26] C.L. Terng, G. Thorbergsson. *Completely integrable flows on adjoint orbits.* Result Math. **40** (2001), 286-309.
[27] C.L. Terng, K. Uhlenbeck. *Schrödinger flows on Grassmannians, Integrable systems, geometry and topology,* AMS/IP Stud. Adv.Math, AMS, Providence (2006), 235-256.
[28] C.L. Terng, K. Uhlenbeck. *Backlund Transformations and Loop Group Actions,* Communications in Pure and Applied Mathematics, **53** (2000), 1-75.
[29] Y. Yasui and N. Sasaki. *Differential geometry of the vortex filament equation,* J. Geom. Phys. **28** 1–2 (1998), 195–207.

309 Van Vleck Hall, UW Madison, WI 53705, USA
E-mail address: maribeff@math.wisc.edu