INTEGRABLE FLOWS FOR STARLIKE CURVES IN CENTROAFFINE SPACE

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In honor of Peter Olver

Abstract. We construct integrable hierarchies of flows for curves in centroaffine $\mathbb{R}^3$ through a natural symplectic structure on the space of closed unparametrized starlike curves. We show that the induced evolution equations for the differential invariants are closely connected with the Boussinesq hierarchy, and prove that the restricted hierarchy of flows on curves that project to conics in $\mathbb{RP}^2$ induces the Kaup-Kuperschmidt hierarchy at the curvature level.

1. Introduction

1.1. **Integrable evolutions of space curves.** Much of the work on integrable curve evolution equations has been guided by the fundamental role played by the differential invariants of the curve (e.g., curvature and torsion in the Euclidean setting) in helping identify the curve evolution as an integrable one. Perhaps the most important example in the case of space curves is that of the Localized Induction Equation (LIE)

$$\gamma_t = \gamma_x \times \gamma_{xx}, \tag{1}$$

describing the evolution of a curve with position vector $\gamma(x,t)$, and arclength parameter x. The complete integrability of equation (1) was uncovered by the realization, due to Hasimoto [8], that the function $\psi = \kappa \exp(\mathrm{i}\int \tau \,\mathrm{d}x)$, of the curvature κ and torsion τ of γ, is a solution of the cubic focusing nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0, \tag{2}$$

one of the two best-known integrable nonlinear wave equations (the other being the KdV equation).

In this paper, we also use as a guiding principle the observation that many (but not all) integrable curve evolutions have the property of *local preservation of arclength*, i.e., the associated vector fields satisfy a *non-stretching* condition. For example, the LIE vector field $W = \gamma_x \times \gamma_{xx}$ satisfies the condition $\delta_W \|\gamma_x\| = 0$, where δ_W denotes the variation in the direction of W. Thus the local arclength parameter x is independent of t, and the compatibility conditions $\gamma_{xt} = \gamma_{tx}, \gamma_{xxt} = \gamma_{txx}, \gamma_{xxxt} = \gamma_{txxx}$ (more commonly written as compatibility conditions of the Frenet equations and the evolution equations for the Frenet frame) turn out to be equivalent to the Lax pair of the NLS equation for ψ.

Indeed, many integrable curve evolutions in various geometries have been found by looking for non-stretching vector fields (in a given geometry) that produce compatible equations for the moving frame of the evolving curve; in the case of space curves, the geometries explored include Euclidean [10], spherical [5], Minkowski [19], affine and centroaffine [4]. (Moreover, integrable curve evolutions without preservation of arclength have been found in projective ([15]), conformal ([16]) and other parabolic manifolds). The approach in these investigations

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involves finding suitable choices for the coefficients of the non-stretching vector fields (relative to a Frenet-type frame) and often assuming special relations among the differential invariants; thus it can be challenging to identify integrable hierarchies.

Another approach to investigating the relation between a non-stretching curve evolution and integrable PDE system for the differential invariants is to seek a natural Hamiltonian setting for the curve flow. The LIE was shown by Marsden and Weinstein [17] to be a Hamiltonian flow on a suitable phase space endowed with a symplectic form of hydrodynamic origin (see also [1, 2]). In a fundamental paper [12] Langer and Perline used this framework to explore in depth the correspondence between the LIE and NLS equations and, along the way, derived a geometric recursion operator *at the curve level* that made it easy to obtain the integrable hierarchies of both curve and curvature flows, as well as meaningful reductions thereof [13, 11].

In this article we study integrable evolution equations for closed curves in centroaffine $\mathbb{R}^3$ beginning, as in [12], with a natural symplectic form on an appropriate infinite-dimensional phase space. The Hamiltonian setting allows us to construct integrable hierarchies of curve flows and the associated families of integrable evolution equations for the centroaffine differential invariants (which turn out to be equivalent to the Boussinesq hierarchies). The motivation for addressing the centroaffine case comes from an interesting article by Pinkall [23], who derived a Hamiltonian evolution equation on the space of closed non-degenerate curves in the centroaffine plane. The simple definition of the symplectic form in the planar case (related to the $SL(2)$-invariant area form) suggested that an analogous description may be possible in the 3-dimensional case, where a parallel could be drawn with the more familiar Euclidean case treated by [12].

Before describing the organization of the paper, we briefly discuss Pinkall's original setting and some results of ours for the planar case.

1.2. **Pinkall's Flow in $\mathbb{R}^2$.** Centroaffine differential geometry in $\mathbb{R}^n$ refers to the study of submanifolds and their properties that are invariant under the action of $\mathrm{SL}(n)$, not including translations. For example, a parametrized curve $\gamma : I \to \mathbb{R}^n$ (where I is an interval on the real line) is *nondegenerate* if

$$\det(\gamma(x), \gamma'(x), \dots, \gamma^{(n-1)}(x)) \neq 0$$

for all $x \in I$, and this property is clearly invariant under the action of $SL(n)$. Thus, for these curves the integral

$$\int |\gamma, \gamma', \dots, \gamma^{(n-1)}|^{1/n}\, \mathrm{d}x \tag{3}$$

is $SL(n)$-invariant, and represents the *centroaffine arclength*, where for the sake of convenience we use the notation

$$|X_1, \dots, X_n| := \det(X_1, \dots, X_n)$$

for n-tuples of vectors $X_i \in \mathbb{R}^n$. (The nth root in (3) is necessary to make the integral invariant under reparametrization.)

In the case where $n = 2$, Pinkall [23] defined a geometrically natural flow for nondegenerate curves in $\mathbb{R}^2$, which he referred to as *star-shaped curves*, as follows. Suppose that γ is parametrized by centroaffine arclength s, so that $|\gamma, \gamma'| = 1$ identically. It follows that $\gamma_{ss} = -p(s)\gamma$ where p is defined as the centroaffine curvature. Along a closed curve γ, one defines the skew-symmetric form

$$\omega(X,Y) = \oint_\gamma |X,Y|\, ds, \tag{4}$$

where X and Y are vector fields along γ. This pairing is nondegenerate on the space of vector fields that locally preserve arclength. Then the symplectic dual with respect to (4) of the functional $\oint_\gamma p(s)\, ds$ is the vector field

$$X = \tfrac{1}{2}p'\gamma - p\gamma_s.$$

Pinkall's flow $\gamma_t = \frac{1}{2}p'\gamma - p\gamma_s$ induces an evolution equation for curvature that coincides with the KdV equation, up to rescaling. In an earlier paper [3], we showed how to use solutions of the (scalar) Lax pair for KdV to generate solutions of Pinkall's flow. In particular, we showed that varying the spectral parameter in the Lax pair for a fixed KdV potential q corresponds to constructing a solution to the flow with curvature given by a Galileian KdV symmetry applied to q. We also derived conditions under which periodic KdV solutions corresponded to smoothly closed loops (for appropriate values of the spectral parameter) and illustrated this using finite-gap KdV solutions.

1.3. **Organization of the paper.** In §2 we introduce basic notions concerning the differential geometry of curves in centroaffine $\mathbb{R}^3$ (starlike curves), including centroaffine arclength, differential invariants, and non-stretching curve variations. This section also contains a discussion of the relation between starlike curves and parametrized maps into $\mathbb{RP}^2$. In §3 we generalize Pinkall's setting and introduce a symplectic form on the space of closed unparametrized starlike curves; we also compute the Hamiltonian vector fields associated with the total length and total curvature functionals. These Hamiltonian curve flows induce evolution equations for the differential invariants; we discuss these equations in §4, including their bi-Hamiltonian formulation, Lax representation, and the connection with the Boussinesq equation. In §5 we show that the Poisson operators introduced in §4 give rise to the Boussinesq recursion operator, generating a (double) hierarchy of commuting evolution equations for the differential invariants. In Theorem 5.4, we relate the Hamiltonian structure for starlike curves and the Poisson structure for the differential invariants, and obtain a double hierarchy of centroaffine geometric evolution equations. We conclude §5, and the paper, by considering which of these flows preserve the property that γ corresponds to a conic under the usual projectivization map $\pi : \mathbb{R}^3 \to \mathbb{RP}^2$. We show that the sub-hierarchy of conicity-preserving curve evolutions induces the Kaup-Kuperschmidt hierarchy at the curvature level.

2. Centroaffine Curve Flows in $\mathbb{R}^3$

2.1. **Centroaffine Invariants.** Let $\gamma : I \to \mathbb{R}^3$ be nondegenerate. We parameterize γ by centroaffine arclength, so that

$$|\gamma, \gamma', \gamma''| = 1. \tag{5}$$

We assume for the rest of this subsection that x is an arclength parameter.

It follows by differentiating (5) with respect to x that

$$\gamma''' = p_0\gamma + p_1\gamma' \tag{6}$$

for some functions $p_0(x)$ and $p_1(x)$. As explained below, these constitute a complete set of differential invariants for nondegenerate curves.

Remark 2.1. Some insight into the meaning of the centroaffine curve invariants can be gained by considering the relationship between γ and the corresponding parametrized curve $\Upsilon = \pi \circ \gamma$ in $\mathbb{RP}^2$, where $\pi : \mathbb{R}^3 \to \mathbb{RP}^2$ is projectivization. The nondegeneracy condition on γ corresponds to Υ being regular and free of inflection points. Conversely, any such parametrized curve $\Upsilon : \mathbb{R} \to \mathbb{RP}^2$ has a unique lift to $\gamma : \mathbb{R} \to \mathbb{R}^3$ which is centroaffine arclength-parametrized; we refer to γ as the *canonical lift* of Υ. When written in terms of Υ instead of γ, the invariants p_0 and p_1 are (up to sign) the well-known Wilczynski invariants [27]. Since these invariants define a differential equation whose solution determines the curve uniquely up to the action of the group $SL(3)$, any other differential invariant must be functionally dependent on p_0, p_1 and their x-derivatives.

According to Ovseinko and Tabachnikov [22], the cubic differential $(p_0 - \frac{1}{2}p_1')(\mathrm{d}x)^3$ has the interesting property that it is invariant under reparametrizations of Υ. In fact, those curves in $\mathbb{RP}^2$ for which this differential vanishes identically are conics. For curves for which the coefficient $p_0 - \frac{1}{2}p_1'$ is nowhere vanishing, one can define the *projective arclength* differential $(p_0 - \frac{1}{2}p_1')^{1/3}\mathrm{d}x$. Those parametrized curves in $\mathbb{RP}^2$ for which $p_0 - \frac{1}{2}p_1' = C$, where C is a nonzero constant, are parametrized proportional to projective arclength, and we use the same terminology for their canonical lifts in $\mathbb{R}^3$. (Note that, in this case, the projective arclength differential is $C^{1/3}$ times the centroaffine arclength differential.)

Remark 2.2. Huang and Singer [9] refer to nondegenerate curves in centroaffine $\mathbb{R}^3$ as *star-like*. They define invariants κ and τ which correspond to $-p_1$ and p_0 respectively. Labeling p_0 as torsion is appropriate, since nondegenerate curves that lie in a plane in $\mathbb{R}^3$ (not containing the origin) are exactly those for which p_0 is identically zero.

Along a nondegenerate curve, an analogue of the Frenet frame is provided by vectors $\gamma, \gamma', \gamma''$. In fact, if we combine them as columns in an $SL(3)$-valued matrix $W = (\gamma, \gamma', \gamma'')$, then the analogue of the Frenet equations is

$$W_x = W \begin{pmatrix} 0 & 0 & p_0 \\ 1 & 0 & p_1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{7}$$

However, for later use it will be convenient to define a different SL(3)-valued frame $F(x) = (\gamma, \gamma', \gamma'' - p_1\gamma)$ which satisfies the Frenet-type equation

$$F_x = FK, \qquad K = \begin{pmatrix} 0 & k_1 & k_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{8}$$

where

$$k_1 = p_1, \qquad k_2 = p_0 - p_1'. \tag{9}$$

Of course, k_1, k_2 also constitute a complete set of differential invariants, and we will come to use these in place of the Wilczynski invariants from §4 onwards.

2.2. **Nonstretching Variations.** Suppose that $\Gamma : I \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ is a smooth mapping such that, for fixed t, $\Gamma(x,t)$ is a nondegenerate curve parametrized by x. Without loss of generality, we will assume that $\gamma(x) = \Gamma(x, 0)$ is parametrized by centroaffine arclength. Let X denote the variation of γ in the t-direction, and expand

$$X = \frac{\partial}{\partial t}\Big|_{t=0} \Gamma = a\gamma + b\gamma' + c\gamma''. \tag{10}$$

(We will use primes to denote derivatives with respect to x; note that x is not necessarily an arclength parameter along the curves in the family for $t \neq 0$.)

To compute the variation of the arclength differential $|\gamma, \gamma', \gamma''|^{1/3}\,\mathrm{d}x$, we introduce the notation δ for the variation in the t-direction. Using the relation (6), we compute

$$\begin{aligned}\delta\gamma' = X' =& (a' + p_0 c)\gamma + (a + b' + p_1 c)\gamma' + (b + c')\gamma'',\\ \delta\gamma'' = X'' =& (a'' + 2p_0 c' + p_0' c + p_0 b)\gamma + (2a' + p_0 c + b'' + 2p_1 c' + p_1' c + p_1 b)\gamma'\\ &+ (a + 2b' + p_1 c + c'')\gamma''.\end{aligned}$$

Then

$$\delta|\gamma, \gamma', \gamma''| = |\delta\gamma, \gamma', \gamma''| + |\gamma, \delta\gamma', \gamma''| + |\gamma, \gamma', \delta\gamma''| = 3a + 3b' + c'' + 2p_1 c.$$

In particular, the variation X preserves the centroaffine arclength differential if and only if

$$b' = -a - \tfrac{1}{3}(c'' + 2p_1 c), \tag{11}$$

i.e.,

$$X = a\gamma - \left(\int \left(a + \tfrac{1}{3}c'' + \tfrac{2}{3}p_1 c\right) \mathrm{d}x\right)\gamma' + c\gamma''. \tag{12}$$

We refer to vector fields of this form as *non-stretching*, since not only do such variations preserve the overall arclength of, say, a closed loop, but also no small portion of the curve is stretched or compressed.

3. Hamiltonian Curve Flows

3.1. **Symplectic Structure on Starlike Loops.** Generalizing Pinkall's setting [23] for planar star-shaped loops to the three-dimensional case, we introduce the infinite-dimensional space

$$\widehat{M} = \{\gamma : S^1 \to \mathbb{R}^3 \;:\; |\gamma, \gamma', \gamma''| = 1\},$$

as a subset of the vector space $V = \mathrm{Map}(S^1, \mathbb{R}^3)$ of $\mathcal{C}^\infty$ maps from S^1 to $\mathbb{R}^3$. Assume that $\gamma \in \widehat{M}$, i.e., γ is a starlike curve parametrized by centroaffine arclength; then a vector field $X = a\gamma + b\gamma' + c\gamma''$ is in the tangent space $T_\gamma\widehat{M}$ if and only if X is of the form (12).

On V define the skew-symmetric form

$$\omega_\gamma(X, Y) = \oint_\gamma |X, \gamma', Y|\,\mathrm{d}x, \qquad X, Y \in T_\gamma V. \tag{13}$$

Note that ω is automatically closed (that is, $\mathrm{d}\omega = 0$) since the integrand in (13) is a volume form on $\mathbb{R}^3$ [1, 2].

Letting $X = a\gamma + b\gamma' + c\gamma''$, $Y = \tilde{a}\gamma + \tilde{b}\gamma' + \tilde{c}\gamma''$, we compute

$$\omega_\gamma(X, Y) = \oint_\gamma (a\tilde{c} - \tilde{a}c)\,\mathrm{d}x. \tag{14}$$

Thus, $\omega_\gamma(X, Y) = 0$ for every Y if and only if $a = c = 0$. By setting $a = c = 0$ in (12), we see that the kernel of the restriction of ω to $\widehat{M}$ consists of constant multiples of γ'. Therefore, we define the quotient space $M = \widehat{M}/\mathbb{R}$, where the action of $\mathbb{R}$ takes $\gamma(x)$ to $\gamma(x + d)$. By construction, M is the space of unparametrized starlike loops, and ω descends to give a well-defined symplectic form on M.

3.2. **Examples.** Recall that the correspondence between vector fields X_H and (differentials of) Hamiltonians H on a symplectic manifold M is defined using the symplectic form by the relation

$$\mathrm{d}H[X] = \omega_\gamma(X, X_H), \quad \forall X \in T_\gamma M. \tag{15}$$

We will use this to compute the Hamiltonian vector fields for a few interesting functionals; note that these vector fields are unique only up to adding a constant times γ'.

We first consider the arclength functional

$$L(\gamma) = \oint_\gamma |\gamma, \gamma', \gamma''|^{1/3} \mathrm{d}x. \tag{16}$$

Given an arbitrary vector field $X = a\gamma + b\gamma' + c\gamma''$ (not necessarily arclength preserving), the variation of the determinant in (16) along X is given by

$$\delta|\gamma, \gamma', \gamma''| = |X, \gamma', \gamma''| + |\gamma, X', \gamma''| + |\gamma, \gamma', X''| = (3a + 3b' + c'' + 2p_1 c)|\gamma, \gamma', \gamma''|.$$

Assuming that $|\gamma, \gamma', \gamma''|$=1 and setting all constants of integration to zero, we obtain

$$\mathrm{d}L[X] = \oint_\gamma (a + \tfrac{2}{3} p_1 c)\, \mathrm{d}x.$$

Suppose that $X_L = \tilde{a}\gamma + \tilde{b}\gamma' + \tilde{c}\gamma''$. Then setting $\mathrm{d}L[X] = \omega_\gamma(X, X_L) = \oint (a\tilde{c} - \tilde{a}c)\mathrm{d}x$, and using the non-stretching condition (11), we obtain the following Hamiltonian vector field on M:

$$X_L = \gamma'' - \tfrac{2}{3} p_1 \gamma. \tag{17}$$

In §4.4 we will see that the associated curve flow $\gamma_t = X_L$ leads to the Boussinesq equation for the curvatures k_1, k_2.

Next, we introduce the total curvature functional

$$P(\gamma) = \oint p_1\, \mathrm{d}x. \tag{18}$$

From $\gamma''' = p_0\gamma + p_1\gamma'$ and (5), it follows that $p_1 = |\gamma, \gamma''', \gamma''|$. Then the variation of p_1 along an arbitrary vector field $X = a\gamma + b\gamma' + c\gamma''$ is given by

$$\begin{aligned} \delta p_1 =& |X, \gamma''', \gamma''| + |\gamma, X''', \gamma''| + |\gamma, \gamma''', X''| \\ =& |X, p_0\gamma + p_1\gamma', \gamma''| + |\gamma, X''', \gamma''| + |\gamma, p_0\gamma + p_1\gamma', X''| \\ =& 3p_1 a + 3c' p_0 + 2c p_0' + 3a'' + b''' + 4c'' p_1 + 3c' p_1' + c p_1'' + 5b' p_1 + b p_1' + 2c p_1^2. \end{aligned} \tag{19}$$

Then, up to perfect derivatives,

$$\mathrm{d}P[X] = \oint \delta p_1\, \mathrm{d}x = \oint 3p_1 a + 3c' p_0 + 2c p_0' + 4c'' p_1 + 3c' p_1' + c p_1'' + 4b' p_1 + 2c p_1^2\, \mathrm{d}x.$$

Assuming X is an arclength-preserving vector field, we set $b' = -a - \frac{1}{3}(c'' + 2p_1 c)$ and compute

$$\oint \delta p_1\, \mathrm{d}x = \oint -a p_1 + 3c' p_0 + 2c p_0 + \tfrac{8}{3} c'' p_1 + 3c' p_1' + c p_1'' - \tfrac{2}{3} c p_1^2\, \mathrm{d}x.$$

Integrating by parts, we arrive at

$$\mathrm{d}P(X) = \oint -p_1 a + (-p_0' + \tfrac{2}{3} p_1'' - \tfrac{2}{3} p_1^2) c\, \mathrm{d}x. \tag{20}$$

Suppose that $X_P = \tilde{a}\gamma + \tilde{b}\gamma' + \tilde{c}\gamma''$. Setting the right-hand side of (20) equal to $\omega_\gamma(X, X_P) = \oint (a\tilde{c} - \tilde{a}c)\, \mathrm{d}x$, we get $\tilde{a} = p_0' - \frac{2}{3} p_1'' + \frac{2}{3} p_1^2$ and $\tilde{c} = -p_1$. Using equation (11) we compute

$\tilde{b}' = -(p_0' - \frac{2}{3}p_1'' + \frac{2}{3}p_1^2) - \frac{1}{3}(-p_1'' - 2p_1^2) = (p_1' - p_0)'$, a perfect derivative. Thus the Hamiltonian vector field corresponding to (18) is

$$X_P = \left(\tfrac{2}{3}(p_1^2 - p_1'') + p_0'\right)\gamma + (p_1' - p_0)\gamma' - p_1\gamma''.$$

Again, we will see that the associated curve flow $\gamma_t = X_P$ is also directly related to one of the flows in the Boussinesq hierarchy of equations for the curvatures of γ.

4. Integrable Centroaffine Curve Flows

In this section we will examine the evolution of centroaffine curvatures induced by the Hamiltonian curve flows defined in §3.2. We begin by computing the evolution of invariants under more general curve flows.

4.1. **Evolution of Invariants.**

First, we consider how the centroaffine invariants of a starlike curve evolve under a general nonstretching evolution equation

$$\gamma_t = r_0\gamma + r_1\gamma' + r_2\gamma'' \tag{21}$$

with $r_0 = -r_1' - \frac{1}{3}(r_2'' + 2p_1r_2)$. From now on, we assume that γ is parametrized by affine arclength x at each time. Of course, in order for (21) to represent a *geometric* evolution equation, r_1 and r_2 should be functions of the invariants p_0, p_1 and their arclength derivatives.

Proposition 4.1. *The evolution equations induced by* (21) *for the Wilzcynski invariants are*

$$\begin{aligned}(p_0)_t =& - r_1'''' + p_1r_1'' + 3p_0r_1' + p_0'r_1 \\ & - \tfrac{1}{3}(r_2''''' + p_1r_2''') + ((3p_0 - 2p_1')r_2')' + \tfrac{2}{3}(p_1^2r_2' + (p_1p_1' - p_1''')r_2) + p_0''r_2 \end{aligned}\tag{22}$$

$$(p_1)_t = - 2r_1''' + 2p_1r_1' + p_1'r_1 - r_2'''' + p_1r_2'' + (3p_0 - p_1')r_2' + (2p_0' - p_1'')r_2 \tag{23}$$

Proof. The second equation (23) follows by substituting $a = -r_1' - \frac{1}{3}(r_2'' + 2p_1r_2), c = r_2$ in the last line of (19). Similarly, using $p_0 = |\gamma''', \gamma', \gamma''|$, we compute

$$(p_0)_t = |(\gamma_t)''', \gamma', \gamma''| + |p_0\gamma + p_1\gamma', (\gamma_t)', \gamma''| + p_0|\gamma, \gamma', (\gamma_t)''|$$

and obtain (22). □

From now on, we will take $k_1 = p_1$ and $k_2 = p_0 - p_1'$ as fundamental invariants; one reason for doing this is that the evolution equations for these invariants induced by (21) take the form

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \tag{24}$$

where $\mathcal{P}$ is the skew-adjoint matrix differential operator

$$\mathcal{P} = \begin{pmatrix} -2D^3 + Dk_1 + k_1D & -D^4 + D^2k_1 + 2Dk_2 + k_2D \\ D^4 - k_1D^2 + 2k_2D + Dk_2 & \frac{2}{3}(D^5 + k_1Dk_1 - k_1D^3 - D^3k_1) + [k_2, D^2] \end{pmatrix} \tag{25}$$

and D stands for the derivative with respect to x and $[\cdot,\cdot]$ denotes the commutator on pairs of operators.[1] This operator $\mathcal{P}$, which arises naturally when using k_1, k_2 instead of p_0, p_1, will play a significant role in the integrable structure of the flows we study.

[1]Note that expressions like Dk_1 and Dk_2 denote composition of D with multiplication by k_1 and k_2, respectively.

4.2. **Two Integrable Flows.** The Hamiltonian vector field X_L induces a nonstretching evolution equation

$$\gamma_t = \gamma'' - \tfrac{2}{3}k_1\gamma. \tag{26}$$

(This will be the first non-trivial curve evolution in the hierarchy discussed in §5, where the right-hand side is labeled as Z_1.) By setting $r_1 = 0$, $r_2 = 1$ in (24), we obtain the corresponding curvature evolution

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k_1'' + 2k_2' \\ \tfrac{2}{3}(k_1k_1' - k_1''') + k_2'' \end{pmatrix}. \tag{27}$$

This PDE system for curvatures is itself Hamiltonian, since it can be written in the form

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\mathsf{E}k_2,$$

where E denotes the vector-valued Euler operator

$$\mathsf{E}f = \left(\sum_{j\geq 0}(-D)^j\frac{\partial f}{\partial k_1^{(j)}}, \sum_{j\geq 0}(-D)^j\frac{\partial f}{\partial k_2^{(j)}}\right)^T \tag{28}$$

on scalar functions f of k_1, k_2 and their higher x-derivatives $k_1^{(j)}, k_2^{(j)}$. (Technically, the Poisson bracket defined using the Hamiltonian operator $\mathcal{P}$ on the appropriate function space—see §5.2 below—must satisfy the usual requirements of skew-symmetry and the Jacobi identity.)

Moreover, (27) can also be written in Hamiltonian form as

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{Q}\,\mathsf{E}\rho_3, \tag{29}$$

for a different Hamiltonian operator and density

$$\mathcal{Q} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \qquad \rho_3 := \tfrac{1}{3}(k_1')^2 + k_2k_1' + k_2^2 - \tfrac{1}{9}k_1^3. \tag{30}$$

(The notation ρ_3 is explained below.) Since the curvature evolution can be written in Hamiltonian form in two ways (27) and (29), the integrals $\int k_2\,\mathrm{d}x$ and $\int \rho_3\,\mathrm{d}x$ are conserved by the flow (for appropriate boundary conditions).

Remark 4.2. In fact, the curvature evolution here is a *bi-Hamiltonian system*, because $\mathcal{P}$ and $\mathcal{Q}$ are a *Hamiltonian pair*, i.e., their linear combinations form a pencil of Hamiltonian operators, and a pencil of compatible Poisson structures. This assertion can be verified mechanically (see, e.g., section 7.1 in [20] for details), but it also follows from the fact that, at least in the periodic case, the Poisson structures are reductions of a well-known compatible pencil of Poisson brackets on the space of loops in $\mathfrak{sl}(3)$. (Indeed, when γ is periodic, the matrix K in (8) provides a lift into this loop space.) The proof of the reduction of these brackets to the space of differential invariants can be found in [14], where it is shown that $\mathcal{P}$ is associated to the reduction of the Adler-Gel'fand-Dikii bracket for $SL(3)$ and $\mathcal{Q}$ is associated to its companion; see [14] for more details.

The (negative of the) Hamiltonian vector field X_P of §3.2 induces the nonstretching evolution

$$\gamma_t = k_1\gamma'' + k_2\gamma' + r_0\gamma, \tag{31}$$

where

$$r_0 = -(k_2' + \tfrac{1}{3}((k_1'' + 2k_1^2)). \tag{32}$$

(The right-hand side of (31) is labeled as Z_2 in the hierarchy discussed in §5.) We similarly obtain the curvature evolution equations induced by this flow by setting $r_1 = k_2$, $r_2 = k_1$ in (24). We remark that the resulting system is also bi-Hamiltonian, since it can be written as

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\,\mathsf{E}\rho_2 = \mathcal{Q}\,\mathsf{E}\rho_4 \tag{33}$$

for

$$\rho_2 = k_1 k_2, \qquad \rho_4 = \tfrac{1}{3}(k_1'')^2 + k_1''(k_2' - k_1^2) - k_1(k_1')^2.$$

Thus, $\int \rho_2\,\mathrm{d}x$ and $\int \rho_4\,\mathrm{d}x$ are conserved integrals for (31). (Because X_P corresponds symplectically to the Hamiltonian $\int k_1\,\mathrm{d}x$, it is automatic that this integral is also conserved.)

Remark 4.3. The arclength normalization (5) is preserved by the simultaneous rescaling $x \mapsto \lambda x$, $\gamma \mapsto \lambda^{-1}\gamma$. Under this rescaling, k_1 and k_2 scale by λ^2 and λ^3 respectively. Thus, we may assign *scaling weights* 2 and 3 respectively to these curvatures, with weight increasing by one for every x-derivative.

It will turn out (see §5.1 below) that the conserved densities for evolution equations (26) and (31) are all of homogeneous weight, with one density for each positive weight not congruent to 1 modulo 3. We will number the densities in order of increasing weight, letting $\rho_0 = k_1$, $\rho_1 = k_2$ and so on; the density in (29) is denoted by ρ_3, since its weight falls between those of ρ_2 and ρ_4.

The curve flows (26) and (31) turn out to share the same conservation laws; for example, $\int k_1\,\mathrm{d}x$ is conserved by (26) because (27) implies that

$$(k_1)_t = D(k_1' + 2k_2).$$

Similarly, (31) conserves $\int k_2\,\mathrm{d}x$ because (33) implies that

$$(k_2)_t = D\left(\tfrac{2}{3}k_1^{(4)} + k_2''' - 2k_1k_1'' - (k_1')^2 - 2k_1k_2' + \tfrac{4}{9}k_1^3 + 2k_2^2\right).$$

In §5 we will show that these flows share an infinite sequence of conservation laws.

4.3. **Lax Representation.** In this subsection we use geometric considerations to derive Lax pairs for curvature evolution equations induced by (26) and (31).

In [3], we found that the components of the solution $\gamma(x,t)$ of Pinkall's flow satisfied the scalar Lax pair for the KdV equation. In the same spirit, we seek a system of the form

$$\mathcal{L}y = 0, \qquad y_t = \mathcal{M}y, \tag{34}$$

satisfied by each component of γ, where $\mathcal{L}$ and $\mathcal{M}$ are differential operators in x with coefficients involving k_1, k_2. Using (6), we see that every component of γ satisfies the scalar ODE $y''' = (k_1 y)' + k_2 y$, and so we will let

$$\mathcal{L} := D^3 - Dk_1 - k_2$$

and seek operators $\mathcal{M}_1$ for (26) and $\mathcal{M}_2$ for (31).

In the case of (26), the components of γ also satisfy $y_t = y'' - \frac{2}{3}k_1 y$, so we choose

$$\mathcal{M}_1 := D^2 - \tfrac{2}{3}k_1.$$

One can then verify that (27) implies that

$$\mathcal{L}_t = [\mathcal{M}_1, \mathcal{L}]. \tag{35}$$

In the case of (31), the components of γ satisfy $y_t = k_1 y'' + k_2 y' + r_0 y$, with r_0 as given by (31). So, we might set $\mathcal{M}_2 = k_1 D^2 + k_2 D + r_0$. However, (34) would also be satisfied if we modify $\mathcal{M}_2$ by adding $\mathcal{N}\mathcal{L}$, where $\mathcal{N}$ is an arbitrary differential operator. In fact, the system (33) actually implies that $\mathcal{L}_t = [\mathcal{M}_2, \mathcal{L}]$ for

$$\mathcal{M}_2 := (k_1 D^2 + k_2 D + r_0) - 3D\mathcal{L}.$$

Writing these systems in Lax form enables us to interpolate a spectral parameter into the linear equations satisfied by the components. Thus, consider solutions of the compatible system

$$\mathcal{L}y = \lambda y, \qquad y_t = \mathcal{M}_j y, \tag{36}$$

where $j = 1$ or $j = 2$. Of course, the components of the evolving curve satisfy (36) only when $\lambda = 0$. When $\lambda \neq 0$, we can construct solutions of the curve flow using solutions of (34):

Proposition 4.4. *Let k_1, k_2 satisfy the evolution equation* (27) *for $j = 1$ or* (33) *for $j = 2$. For fixed $\lambda \in \mathbb{R}$, let y_1, y_2, y_3 be linearly independent solutions of* (36)*, with Wronskian W. Then W is constant in x and t, and $\gamma = W^{-1/3}(y_1, y_2, y_3)^T$ is arclength-parametrized at each time t, with centroaffine invariants k_1 and $\tilde{k}_2 = k_2 + \lambda$. Furthermore, γ satisfies the evolution equation*

$$\gamma_t = \begin{cases} \gamma'' - \frac{2}{3}k_1\gamma, & j = 1, \\ k_1\gamma'' + (\tilde{k}_2 - 4\lambda)\gamma' + r_0\gamma, & j = 2. \end{cases}$$

Proof. If we let $\mathsf{y} = (y_1, y_2, y_3)^T$ and form the matrix $F = (\mathsf{y}, \mathsf{y}', \mathsf{y}'')$, then F satisfies differential equations of the form

$$F^{-1}F_x = \begin{pmatrix} 0 & 0 & k_2 + k_1' + \lambda \\ 1 & 0 & k_1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad F^{-1}F_t = N_j,$$

where both right-hand side matrices have trace zero. For example, when $j = 1$ one can directly calculate, by differentiating $y_t = \mathcal{M}_1 y$, that

$$N_1 = \begin{pmatrix} -\frac{2}{3}k_1 & k_2 + \frac{1}{3}k_1' + \lambda & k_2' + \frac{1}{3}k_1'' \\ 0 & \frac{1}{3}k_1 & k_2 + \frac{2}{3}k_1' + \lambda \\ 1 & 0 & \frac{1}{3}k_1 \end{pmatrix}.$$

Thus, the Wronskian W is constant in x and t.

Because $\gamma''' = (k_1\gamma)' + (k_2 + \lambda)\gamma$, the centroaffine invariants of γ are k_1 and $\tilde{k}_2$. It is straightforward to compute γ_t in the $j = 1$ case, using $y_t = \mathcal{M}_1 y$. In the $j = 2$ case, we compute

$$y_t = \mathcal{M}_2\, y = k_1 y'' + (\tilde{k}_2 - \lambda)y' + r_0 y - 3D\mathcal{L}y = k_1 y'' + (\tilde{k}_2 - 4\lambda)y' + r_0 y.$$

□

4.4. **Connection with Boussinesq Equations.** In [4], Chou and Qu note that, under the centroaffine curve flow (26), the curvatures k_1, k_2 satisfy a two-component system of evolution equations that is equivalent to the Boussinesq equation. This suggests that the other integrable flow (31) under discussion may be related to the Boussinesq hierarchy.

Dickson *et al* [7] write the (first) Boussinesq equation as a system

$$\begin{aligned} (q_0)_t + \frac{1}{6}q_1''' + \frac{2}{3}q_1 q_1' &= 0 \\ (q_1)_t - 2q_0' &= 0 \end{aligned} \tag{37}$$

They embed this in a hierarchy of integrable equations, each of which is written in Lax form as

$$L_t = [P_m, L], \qquad L := D^3 + q_1 D + \tfrac{1}{2}q_1' + q_0, \tag{38}$$

where P_m is a differential operator of order $m \not\equiv 0 \mod 3$, with coefficients depending on q_0, q_1 and their x-derivatives. Note that P_m must be chosen so that $[P_m, L]$ has order one. For example, while $P_1 = D$ yields the trivial evolution $(q_1)_t = q_1'$, $(q_0)_t = q_0'$, setting $P_2 = D^2 + \frac{2}{3}q_1$ gives the Boussinesq equation (37).

Given the resemblance between (35) and (38), it is tempting to find substitutions to connect the Boussinesq equation with (27). In fact, we can make L and $\mathcal{L}$ coincide by setting

$$k_1 = -q_1, \qquad k_2 = \tfrac{1}{2}q_1' - q_0. \tag{39}$$

With this substitution, $\mathcal{M}_1$ coincides with $\mathcal{P}_2$, so it follows that (27) and (37) are equivalent.

In [7] it is shown how the coefficients of the operators P_m can be obtained solving a recursive system of differential equations, and thus these depend on a number of constants of integration. For example, the expression for P_4 is

$$\begin{aligned} P_4 = &\left[f_1 D^2 + (g_1 - \tfrac{1}{2}f_{1x})D + (\tfrac{1}{6}f_1'' - g_1' + \tfrac{2}{3}q_1 f_1)\right] + \\ &+ \left[f_0 D^2 + (g_0 - \tfrac{1}{2}f_0')D + (\tfrac{1}{6}f_0'' - g_0' + \tfrac{2}{3}q_1 f_0)\right] L_3 + k_{4,0} + k_{4,1}L, \end{aligned}$$

where $f_0 = 0, g_0 = 1, f_1 = \frac{1}{3} + c_1, g_1 = \frac{1}{3}q_0 + d_1$, and $k_{4,0}, k_{4,1}, c_1, d_1$ are arbitrary constants. For convenience, we will set all these arbitrary constant to zero, so that

$$P_4 = D^4 + \tfrac{4}{3}q_1 D^2 + \tfrac{4}{3}(q_1' + q_0)D + \tfrac{5}{9}q_1'' + \tfrac{2}{3}q_0' + \tfrac{2}{9}q_1^2. \tag{40}$$

Again, if we use the substitutions (39), we find that the operator $\mathcal{M}_2$ coincides with $-3P_4$. Thus, (33) is equivalent to the second nontrivial flow in the Boussinesq hierarchy, provided we also rescale time by $t \to -3t$.

5. Hierarchies

In [20] the Boussinesq hierarchy is discussed as an example of a biHamiltonian system, in which two sequences of commuting flows (and conservation laws) are generated by applying recursion operators. Thus, given the equivalences established in §4.4, it is not surprising that the Poisson operators of §4 can be combined to give a recursion operator that generates a double hierarchy of commuting evolution equations for k_1, k_2. In fact, we will show that our recursion operator is equivalent to the recursion operator given in [20] (see Example 7.28). The new information we add is that each of these evolution equations is induced by a centroaffine geometric evolution equation for curves, which is itself Hamiltonian relative to the symplectic structure defined in §3.1 (see Theorem 5.4 below).

5.1. **Recursion Operators.** We define a sequence of evolution equations for k_1, k_2

$$\frac{\partial}{\partial t_j}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = F_j[k_1, k_2], \tag{41}$$

via the recursion

$$F_{j+2} = \mathcal{P}\mathcal{Q}^{-1}F_j, \tag{42}$$

with initial data given by

$$F_0 = \begin{pmatrix} k_1' \\ k_2' \end{pmatrix}, \qquad F_1 = \begin{pmatrix} k_1'' + 2k_2' \\ \frac{2}{3}(k_1 k_1' - k_1''') + k_2'' \end{pmatrix}. \tag{43}$$

(Note that F_1 is the right-hand side of (27), while for $j = 0$ (41) gives a simple transport equation for k_1, k_2, corresponding to flow in the direction of the tangent vector γ'.)

In order to assert that the F_j defined by (42) are local functions of k_1, k_2 and their derivatives—i.e., in calculating each F_j, the operator D^{-1} is only applied to exact x-derivatives of local functions—we cite well-known results on the Boussinesq hierarchy. For example, the version of the first Boussinesq equation used by Olver [20] is

$$u_\tau = v', \qquad v_\tau = \tfrac{1}{3}u''' + \tfrac{8}{3}uu', \tag{44}$$

where τ is the time variable. If one considers linear transformations on the variables, it is necessary to use some imaginary coefficients to make our version (27) of the first Boussinesq equation for k_1, k_2 equivalent to (44):

$$x = x, \qquad \tau = \mathrm{i}t, \qquad k_1 = -2u, \qquad k_2 = u' - \mathrm{i}v. \tag{45}$$

Proposition 5.1. *Under the above change of variables, the recursion operator $\mathcal{P}\mathcal{Q}^{-1}$ is equivalent to the Boussinesq recursion operator in* [20].

Proof. The transformation between k_1, k_2 and u, v can be written as

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \mathcal{G}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathcal{G} := \begin{pmatrix} -2 & 0 \\ D & -\mathrm{i} \end{pmatrix}.$$

Thus, if $\partial/\partial t\,(u,v)^T = F[u,v]$ is an evolution equation for u, v, the right-hand side of the corresponding evolution for k_1, k_2 is $\mathcal{G} \circ F$. Thus, our recursion operator $\mathcal{P}\mathcal{Q}^{-1}$ for flows on the k_1, k_2 variables corresponds to a recursion operator

$$\mathcal{G}^{-1}\mathcal{P}\mathcal{Q}^{-1}\mathcal{G} \tag{46}$$

on flows in the u, v variables. In fact, when one calculates (46) and substitutes for k_1, k_2 in terms of u, v using (45), the result is exactly $-\mathrm{i}$ times the Boussinesq recursion operator given in [20]. □

Since in [24] (see section 5.4) it is proven that the Boussinesq recursion operator from [20] always produces local flows when applied to the 'seed' evolution equations (i.e., the tangent flow and first Boussinesq), it follows that the same is true for our recursion operator.

Remark 5.2. Once one checks that the evolution equations (41) for $j = 0$ and $j = 1$ commute, it is automatic from the bi-Hamiltonian structure that all evolution equations in the sequence (41) commute in pairs (see, e.g., Theorem 7.24 in [20]).

It is easy to check that the 'seeds' F_0, F_1 for the recursion are related to the initial conserved densities by

$$F_0 = \mathcal{P}\mathsf{E}\rho_0 = \mathcal{Q}\mathsf{E}\rho_2, \qquad F_1 = \mathcal{P}\mathsf{E}\rho_1 = \mathcal{Q}\mathsf{E}\rho_3. \tag{47}$$

(The second set of equations was derived in §4.2.) While $\mathcal{P}\mathcal{Q}^{-1}$ is the recursion operator for commuting flows, it is evident from (47) that $\mathcal{Q}^{-1}\mathcal{P}$ should be the recursion operator for conservation law *characteristics* (i.e., the result of applying the Euler operator E to a density). In fact, we may define an infinite sequence of conserved densities by

$$\mathsf{E}\rho_{j+2} = \mathcal{Q}^{-1}\mathcal{P}\mathsf{E}\rho_j, \qquad j \geq 0. \tag{48}$$

The first few densities calculated using this recursion appear in Figure 1.

We now use these densities to define a sequence of flows for centroaffine curves, and relate each of them to a curvature evolution equation in the sequence (41). Namely, if f is any local function of k_1, k_2 and their derivatives, we define

$$X^f := (\mathsf{E}f)_1\gamma' + (\mathsf{E}f)_2\gamma'' + r_0\gamma, \tag{49}$$

where the subscripts indicate the components given by (28) and r_0 is determined by the non-stretching condition. Then for the sequence of densities defined recursively by (48) we define the vector fields

$$Z_j := X^{\rho_j} \tag{50}$$

and the corresponding sequence of curve flows

$$\gamma_t = Z_j. \tag{51}$$

Proposition 5.3. *For each $j \geq 0$ the curve flow (51) induces the evolution $\frac{\partial}{\partial t}(k_1, k_2)^T = F_j$ for curvature.*

Proof. From (47) and the recursion relations, it follows by induction that

$$F_j = \mathcal{P}\mathsf{E}\rho_j, \qquad j \geq 0. \tag{52}$$

Then the result follows immediately from (24). □

5.2. **Hamiltonian structure at the curve level.** We now consider the question of how the Hamiltonian operator $\mathcal{P}$ is related to the Hamiltonian structure defined at the curve level in §3.1. Recall from §3.2 that X_H denotes the Hamiltonian vector field associated to the functional H by the requirement that

$$dH[Y] = \omega_\gamma(Y, X_H)$$

for any non-stretching vector field Y.

Theorem 5.4. *Let $H(\gamma) = \oint_\gamma \rho \, \mathrm{d}x$ and assume that*

$$\mathsf{E}\widehat{\rho} = \mathcal{Q}^{-1}\mathcal{P}\mathsf{E}\rho, \tag{53}$$

i.e., $\widehat{\rho}$ is next after ρ in the sequence of densities generated by the recursion operator $\mathcal{Q}^{-1}\mathcal{P}$. Then

$$X_H = -X^{\widehat{\rho}}.$$

Proof. Based on the definition (13) of ω, we need to show that

$$\mathrm{d}H[Y] = \oint_\gamma |X^{\widehat{\rho}}, \gamma', Y| \, \mathrm{d}x \quad \forall Y \in T_\gamma M.$$

If $X = a\gamma + b\gamma' + c\gamma''$ and $Y = \tilde{a}\gamma + \tilde{b}\gamma' + \tilde{c}\gamma''$, then from (14),

$$\oint_\gamma |X, \gamma', Y| \, \mathrm{d}x = \oint_\gamma (a\tilde{c} - \tilde{a}c) \, \mathrm{d}x.$$

However, using (11) to eliminate a and $\tilde{a}$, we obtain

$$\oint_\gamma |X, \gamma', Y| \, \mathrm{d}x = \oint \left(-b'\tilde{c} + c\tilde{b}' + \tfrac{1}{3}(c\tilde{c}'' - c''\tilde{c})\right) \mathrm{d}x = -\oint (b'\tilde{c} + c'\tilde{b}) \, \mathrm{d}x,$$

where the last equation follows by integration by parts. Thus,

$$\oint_\gamma |X^{\widehat{\rho}}, \gamma', Y| \mathrm{d}x = -\oint_\gamma (\tilde{b}, \tilde{c})^T \mathcal{Q} \, \mathsf{E}\widehat{\rho} \, \mathrm{d}x = -\oint_\gamma (\tilde{b}, \tilde{c})^T \mathcal{P}\mathsf{E}\rho \, \mathrm{d}x,$$

using (53) in the last step. Then, because $\mathcal{P}$ is skew-adjoint,

$$\oint_\gamma |X^{\widehat{\rho}}, \gamma', Y| \, \mathrm{d}x = \oint_\gamma \mathsf{E}\rho \cdot \mathcal{P} \begin{pmatrix} \tilde{b} \\ \tilde{c} \end{pmatrix} \mathrm{d}x.$$

On the other hand, using the properties of the Euler operator we have

$$dH[Y] = \oint_\gamma (\mathsf{E}\rho)_1 \delta_Y k_1 + (\mathsf{E}\rho)_2 \delta_Y k_2 \,\mathrm{d}x,$$

where δ_Y denotes the first variation in the direction of Y. Now using (24) we have

$$dH[Y] = \oint_\gamma \mathsf{E}\rho \cdot \begin{pmatrix} \delta_Y k_1 \\ \delta_Y k_2 \end{pmatrix} = \oint_\gamma \mathsf{E}\rho \cdot \mathcal{P} \begin{pmatrix} \tilde{b} \\ \tilde{c} \end{pmatrix} \mathrm{d}x.$$

This concludes the proof. □

The following corollaries are immediate consequences of the theorem.

Corollary 5.5. *Define the Poisson bracket*

$$\{H, G\} = \oint_\gamma \mathsf{E}h \cdot \mathcal{P}\mathsf{E}g \,\mathrm{d}x,$$

where $G(\gamma) = \oint_\gamma g \,\mathrm{d}x$ and $H(\gamma) = \oint_\gamma h \,\mathrm{d}x$ are functionals on M, and g, h are functions of k_1, k_2 and their derivatives. Then

$$dH[X^g] = -\omega(X_H, X^g) = \{H, G\}$$

and $\omega(X_H, X^g) = 0$ if and only if $\{H, G\} = 0$.

Corollary 5.6. *All the vector fields Z_j, defined by* (50) *are Hamiltonian for $j \geq 1$.*

Corollary 5.7. *A closed curve γ is critical for the functional $H(\gamma) = \oint_\gamma \rho_j \,\mathrm{d}x$ with respect to non-stretching variations if and only if γ is stationary for a constant-coefficient linear combination of Z_{j+2} and Z_0.*

Proof. A curve γ is critical for H if and only if $dH_j[Y] = \omega(Y, X_H) = 0$ for any non-stretching vector field Y. Since $X_H = Z_{j+2}$ (by Theorem 5.4), this condition is satisfied if and only if Z_{j+2} is in the kernel of ω, i.e., Z_{j+2} is a constant multiple of the tangent vector Z_0. Equivalently $Z_{j+2} + cZ_0 = 0$ along γ for some constant c, expressing the fact that γ must be stationary for such linear combination of vector fields.

□

5.3. **Projective Properties.** As stated in Remark 2.1, a centroaffine curve γ projects to give a conic in $\mathbb{RP}^2$ if and only if the Wilczynski invariants satisfy $p_0 - \frac{1}{2}p_1' = 0$. (The corresponding condition in terms of k_1, k_2 is $k_2 + \frac{1}{2}k_1' = 0$.) In this subsection we will investigate which flows in the hierarchy have the property that, if γ projects to a conic at time zero, then it continues to have a conical projection at subsequent times. We will show later that the equation of the conic in homogeneous coordinates is fixed in time. We will also discuss flows that preserve a parametrization that is proportional to projective arclength; in that case, the corresponding condition in terms of curvatures is that $k_2 + \frac{1}{2}k_1'$ is a nonzero constant along the curve.

These investigations are much easier if, instead of k_2, we use an invariant that vanishes precisely when the condition we are investigating holds. Accordingly, we fix a constant C, and define an alternative pair of invariants

$$u = k_1, \qquad v = k_2 + \tfrac{1}{2}k_1' - C. \tag{54}$$

Thus, the curve is a conic if $v = 0$ when $C = 0$, and the curve has a constant-speed parametrization (relative to projective arclength) if $v = 0$ when $C \neq 0$.

Nonstretching vector field	Conserved density
$Z_0 = \gamma'$	$\rho_0 = k_1$
$Z_1 = \gamma'' - \frac{2}{3}k_1\gamma$	$\rho_1 = k_2$
$Z_2 = k_1\gamma'' + k_2\gamma' + \ldots$	$\rho_2 = k_1k_2$
$Z_3 = (k_1' + 2k_2)\gamma'' + \left(\frac{1}{3}k_1^2 - \frac{2}{3}k_1'' - k_2'\right)\gamma' + \ldots$	$\rho_3 = \frac{1}{3}(k_1')^2 + k_1'k_2 + \frac{1}{9}k_1^3 + k_2^2$
$Z_4 = (-k_1''' - 2k_2'' + 2k_1k_1' + 4k_1k_2)\gamma''$ $+(-\frac{1}{3}(k_1^{(5)} + k_2^{(4)}) + 2(k_1k_1''' + k_1'k_1'' + k_1k_2'')$ $-\frac{4}{3}k_2k_1'' - \frac{2}{3}k_1'k_2' - \frac{8}{3}k_1^2(k_1' + k_2) - 4k_2k_2')\gamma' + \ldots$	$\rho_4 = \frac{1}{3}(k_1'')^2 + k_1''(k_2' - k_1^2) - k_1(k_1')^2$ $+(k_2')^2 - k_1^2k_2' + \frac{1}{9}k_1^4 + 2k_1k_2^2$

(The coefficient of γ in the vector fields can be determined by the nonstretching condition.)

FIGURE 1

- The γ' and γ'' coefficients of Z_j match the components of $\mathsf{E}\rho_j$.
- Densities satisfy the recursion relation $\mathsf{E}\rho_{j+2} = Q^{-1}\mathcal{P}\mathsf{E}\rho_j$
- Z_j induces curvature evolution $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \mathcal{P}\mathsf{E}\rho_j = \mathcal{Q}\mathsf{E}\rho_{j+2}$.
- For $j \geq 1$, Z_j is the Hamiltonian vector field for -$\int \rho_{j-2}\,\mathrm{d}x$ (with $\rho_{-1} = -1$).

We will convert the evolution equations in the hierarchy (at the level of the invariants) to these variables. Suppose that a curve flow causes the invariants to evolve by

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = F[k_1, k_2],$$

where F is a vector-valued function of k_1, k_2 and their derivatives. Then the corresponding evolution equation for the alternative invariants is

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{G} \circ F[u, v - \tfrac{1}{2}u' + C], \qquad \mathcal{G} := \begin{pmatrix} 1 & 0 \\ \frac{1}{2}D & 1 \end{pmatrix}.$$

Similarly, if $\mathcal{R}$ is the recursion operator generating the hierarchy of evolution equations for k_1, k_2, then the recursion operator for the corresponding flows on u, v differs by a gauge transformation:

$$\tilde{\mathcal{R}} = \mathcal{G}\mathcal{R}\mathcal{G}^{-1}, \qquad \text{where } \mathcal{G}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}D & 1 \end{pmatrix}.$$

(It is understood that, in $\mathcal{R}$ on the right-hand side, k_1, k_2 are substituted for in terms of u, v.) Specifically, using $\mathcal{R} = \mathcal{P}\mathcal{Q}^{-1}$ as defined by (25), (30), we compute

$$\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_0 + \tilde{F}_0 D^{-1}\begin{pmatrix} 0 & 1 \end{pmatrix} + \tilde{F}_1 D^{-1}\begin{pmatrix} 1 & 0 \end{pmatrix}, \tag{55}$$

where

$$\tilde{\mathcal{R}}_0 = \begin{pmatrix} 3(v+C) & 2(u - 2D^2) \\ \mathcal{N} & 3(v+C) \end{pmatrix}, \quad \tilde{F}_0 = \begin{pmatrix} u' \\ v' \end{pmatrix}, \quad \tilde{F}_1 = \begin{pmatrix} 2v' \\ \frac{2}{3}uu' - \frac{1}{6}u''' \end{pmatrix},$$

and $\mathcal{N}$ is the scalar differential operator $\frac{1}{6}D^4 - \frac{5}{6}uD^2 - \frac{5}{4}u'D + \frac{2}{3}u^2 - \frac{3}{4}u''$. One can check that the vectors $\tilde{F}_0, \tilde{F}_1$ are the time derivatives of u, v corresponding to the 'seeds' Z_0 and Z_1 for the hierarchy of curve flows.

By applying $\tilde{\mathcal{R}}$ to the seeds $\tilde{F}_0, \tilde{F}_1$, one can generate the right-hand sides of the evolution equations in the hierarchy in terms of u and v. Letting $\tilde{F}_j$ denote these vectors, we compute (for example) that

$$\tilde{F}_2 = \tilde{\mathcal{R}}\tilde{F}_0 = \begin{pmatrix} -2v''' + 4(uv)' + 3Cu' \\ \frac{1}{6}u^{(5)} - uu''' - 2u'u'' + \frac{4}{3}u^2u' + (4v+3C)v' \end{pmatrix},$$

$$\tilde{F}_3 = \tilde{\mathcal{R}}\tilde{F}_1 = \begin{pmatrix} \frac{1}{3}u^{(5)} + \frac{5}{3}(u^2u' - uu''') - \frac{25}{6}u'u'' + (10v+6C)v' \\ \frac{1}{3}v^{(5)} - (\frac{5}{6}v + \frac{1}{2}C)u''' + \frac{5}{3}((u^2v)' - uv''' - u''v') - \frac{5}{2}u'v'' + 2Cuu' \end{pmatrix}.$$

Here, when applying D^{-1} to differential polynomials in u, v the constant of integration is zero.

Notice in particular that if $v \equiv 0$, then the bottom component of $\tilde{F}_3 - 3C\tilde{F}_1$ vanishes. Thus, the flow $Z_3 - 3CZ_1$ preserves the condition that v is identically zero. In fact, we can calculate two infinite sequences of evolution equations that preserve this condition; the right-hand sides of these are

$$G_k = \begin{cases} \sum_{j=0}^{k} \binom{k}{j} (-3C)^j \tilde{F}_{2(k-j)} & k \text{ even} \\ \sum_{j=0}^{k} \binom{k}{j} (-3C)^j \tilde{F}_{2(k-j)+1} & k \text{ odd.} \end{cases} \tag{56}$$

While it is routine to verify that any individual curvature evolution equation in these sequences preserves $v \equiv 0$, it is easier to observe that the members of these sequences satisfy the recursion relation

$$G_{k+2} = (\tilde{\mathcal{R}} - 3C)^2 G_k.$$

Then the fact that they all preserve $v \equiv 0$ is a consequence of the following:

Proposition 5.8. *If a curvature evolution $(u_t, v_t)^T = G_k[u,v]$ in this sequence preserves $v \equiv 0$, then so does the evolution $(u_t, v_t)^T = G_{k+2}[u,v]$.*

Proof. We assume that $G = (D\ell_1, D\ell_2)^T$ for local functions ℓ_1, ℓ_2 of u, v and their derivatives. (This form for G is necessary if we are able to apply operator $\tilde{\mathcal{R}}$ to it and produce local functions.)

Within the ring of polynomials in u, v and their derivatives, let $\mathcal{V}$ denote the ideal generated by v, v', v'', etc. By hypothesis, $D\ell_2 \in \mathcal{V}$, and the same is true for ℓ_2.

We compute

$$(\tilde{\mathcal{R}} - 3C)G = \ell_1\tilde{F}_1 + \ell_2\tilde{F}_0 + \begin{pmatrix} 3vD\ell_1 + 2(u - D^2)D\ell_2 \\ \mathcal{N}D\ell_1 + 3vD\ell_2 \end{pmatrix}. \tag{57}$$

Thus, the bottom component of $\tilde{\mathcal{R}}^2 G$ is given by

$$(\mathcal{N} + \tilde{F}_{12}D^{-1})\left(\ell_1\tilde{F}_{11} + \ell_2\tilde{F}_{01} + 3vD\ell_1 + 2(u - D^2)D\ell_2\right)$$
$$+ (3v + \tilde{F}_{02}D^{-1})\left(\ell_1\tilde{F}_{12} + \ell_2\tilde{F}_{02} + \mathcal{N}D\ell_1 + 3vD\ell_2\right)$$

(Here, $\tilde{F}_{j1}$ and $\tilde{F}_{j2}$ denote the top and bottom entries in the vector $\tilde{F}_j$.) The expression in large parentheses on the top line is the top entry of $(\tilde{\mathcal{R}} - 3C)G$. This polynomial lies in $\mathcal{V}$, and the same is true if we apply $\mathcal{N}$ or D^{-1} to it. On the other hand, because $\tilde{F}_{02} = v' \in \mathcal{V}$, the coefficient in front on the second line also vanishes when $v \equiv 0$. ∎

In the special case when $C = 0$, we see that the following curve flows (as defined by (50)) preserve conicity:

$$Z_0, \quad Z_3, \quad Z_4, \quad Z_7, \quad Z_8, \quad Z_{11}, \quad Z_{12}, \quad \ldots \tag{58}$$

However, when $C \neq 0$, some care needs to be taken in matching the evolution equations for u, v that preserve $v \equiv 0$ with the corresponding linear combinations of the curve flows Z_j.

In the proof of Prop. 5.8, we used the fact that if an exact derivative $D\ell$ lies in $\mathcal{V}$, then by choosing the constant of integration equal to zero, the antiderivative ℓ also lies in $\mathcal{V}$. However, if we express $D\ell$ in terms of k_1, k_2 instead of u, v, and then take an antiderivative, a particular constant of integration must be chosen in order to belong in $\mathcal{V}$. Thus, when we compute the kth evolution equation for u, v by using the recursion operator $\tilde{\mathcal{R}}$ (which involves applying D^{-1}), then convert this to an evolution equation for k_1, k_2, and finally try to match it with a curve flow in the hierarchy (51), we get a linear combination of Z_k with lower-order flows of the same parity. For example, if we substitute (54) into $\tilde{F}_2$, and then apply the operator $\mathcal{G}^{-1}$, we get

$$\begin{aligned}\mathcal{G}^{-1}\tilde{F}_2[k_1, k_2 + \tfrac{1}{2}k_1' - C] &= \begin{pmatrix} -k_1'''' - 2k_2''' + 2k_1k_1'' + 4(k_1k_2)' + 2(k_1')^2 - Ck_1' \\ \tfrac{2}{3}k_1''''' + k_2'''' - 2k_1k_1''' - 4k_1'k_1'' + 4k_2k_2' - 2(k_1k_2')' + \tfrac{4}{3}(k_1)^2k_1' - Ck_2' \end{pmatrix} \\ &= F_2 - CF_0.\end{aligned}$$

Thus, $\tilde{F}_2$ is induced by the curve flow $Z_2 - CZ_0$; similarly, $\tilde{F}_3$ is induced by $Z_3 - 2CZ_1$, and so on.

Similarly, when we apply the recursion operator $(\tilde{\mathcal{R}} - 3C)^2$ to generate higher-order flows that preserve $v \equiv 0$, these constants of integration accumulate and change the relatively nice pattern of the coefficients exhibited by (56). Here is what we get when we compute the first few curve evolutions corresponding to the evolutions G_k:

u, v evolution	centroaffine curve flow
G_0	Z_0
G_1	$Z_3 - 5CZ_1$
G_2	$Z_4 - 7CZ_2 + 14C^2Z_0$
G_3	$Z_7 - 11CZ_5 + 44C^2Z_3 - \frac{220}{3}C^3Z_1$
G_4	$Z_8 - 13CZ_6 + 65C^2Z_4 - \frac{455}{3}C^3Z_2 + \frac{455}{3}C^4Z_0$

5.4. **Conical Evolutions and the Kaup-Kuperschmidt Hierarchy.** In this section we examine special properties of the conicity-preserving flows (58) which enable us to connect our hierarchy of centroaffine curve flows in $\mathbb{R}^3$ with the Kaup-Kuperschmidt hierarchy and with curve flows in centroaffine $\mathbb{R}^2$. We begin with the observation that, when restricted to conical curves, the coefficient of γ'' vanishes for as many of the vector fields in (58) as one cares to check. In other words, this coefficient belongs to $\mathcal{V}$, the ideal within the ring of differential polynomials in k_1, k_2 generated by $k_2 + \frac{1}{2}k_1'$ and its derivatives. In fact, this true in general, as shown in the following:

Proposition 5.9. *If $j \equiv 0$ or $j \equiv -1$ modulo 4, then the bottom component of $\mathsf{E}\rho_j$ belongs in $\mathcal{V}$; hence, the γ'' coefficient of X_j vanishes on conical curves.*

Proof. The statement can be verified directly for $j = 0$ and $j = 3$. For higher values, we use the recursion relation between the characteristics, which implies that $\mathsf{E}\rho_{j+4} = (\mathcal{Q}^{-1}\mathcal{P})^2\mathsf{E}\rho_j$. From Prop. 5.8 we know that the bottom component of $\tilde{F}_j = \mathcal{G}\mathcal{P}\mathsf{E}\rho_j$ lies in $\mathcal{V}$. By inserting powers of $\mathcal{G}$ and $\mathcal{G}^{-1}$ into the recursion relation, we get

$$\mathsf{E}\rho_{j+4} = \mathcal{Q}^{-1}\mathcal{P}\mathcal{Q}^{-1}\mathcal{G}^{-1}\mathcal{G}\mathcal{P}\mathsf{E}\rho_j = \mathcal{Q}^{-1}\mathcal{R}\mathcal{G}^{-1}\tilde{F}_j.$$

As in the proof of Prop. 5.8, we can assume that $\tilde{F}_j = (D\ell_1, D\ell_2)^T$ where $\ell_2 \in \mathcal{V}$. Taking $C = 0$ in equation (57), we see that the top entry of $\mathcal{R}\mathcal{G}^{-1}\tilde{F}_j = \mathcal{G}\tilde{\mathcal{R}}\tilde{F}_j$ is

$$\ell_1 \tilde{F}_{11} + \ell_2 \tilde{F}_{01} + 3vD\ell_1 + (u - D^2)D\ell_2, \tag{59}$$

which clearly is in $\mathcal{V}$. (Note that $v = k_2 + \frac{1}{2}k_1'$ here.) Noting the form of $\mathcal{Q}$, we see that the bottom entry of $E\rho_{j+4}$ is D^{-1} applied to (59), so it must also belong to $\mathcal{V}$. □

Next, we make a connection with curve flows in centroaffine $\mathbb{R}^2$ to show that, for the flows in (58), the cone that the curve lies on is preserved by the time evolution.

Proposition 5.10. *If $\gamma(x,t)$ evolves by any of the vector fields in* (58)*, and $\gamma(x,0)$ lies on a cone through the origin in $\mathbb{R}^3$, then $\gamma(x,t)$ lies on the same cone at later times.*

Proof. Using the action of $SL(3)$ we can, without loss of generality, assume that the equation of the cone is $y_1 y_3 - y_2^2 = 0$. We fix a map V from $\mathbb{R}^2$ onto this cone:

$$\mathsf{V}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2^{-1/3}\begin{pmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{pmatrix}.$$

Of course, when we projectivize on each end this gives the *Veronese embedding* of $\mathbb{RP}^1$ as a quadric in $\mathbb{RP}^2$. The scale factor of $2^{-1/3}$ is chosen so that if $X(x)$ is a parametrized curve in $\mathbb{R}^2$ satisfying the centroaffine normalization $|X, X'| = 1$, then $\Gamma = \mathsf{V} \circ X$ satisfies the normalization $|\Gamma, \Gamma', \Gamma''| = 1$. Moreover, if $p(x)$ is the curvature of X, then the invariants of Γ are $k_1 = -4p$ and $k_2 = 2p'$. Finally, if X evolves by the non-stretching flow

$$X_t = rX' - \tfrac{1}{2}r'X, \tag{60}$$

then $\Gamma(x,t) = \mathsf{V} \circ X(x,t)$ satisfies

$$\Gamma_t = r\Gamma' - r'\Gamma.$$

By Prop. 5.9 all the flows in (58), when restricted to conical curves, are of this form, for some choice of differential polynomial r in k_1. For any initial data $\gamma(x,0)$, we can define a curve $X(x,0)$ in $\mathbb{R}^2$ such that $\gamma(x,0) = \mathsf{V} \circ X(x,0)$ and make $X(x,t)$ evolve by (60). Because $\Gamma(x,t) = \mathsf{V} \circ X(x,t)$ satisfies the same initial value problem, then $\gamma(x,t) = \Gamma(x,t)$ at all times, and $\gamma(x,t)$ lies on the cone defined by $y_1 y_3 - y_2^2 = 0$ at all times. □

In [4], Chou and Qu discovered a non-stretching flow for curves in centroaffine $\mathbb{R}^3$ which preserves the conicity condition $k_2 + \frac{1}{2}k_1' = 0$ and which causes the curvature k_1 to evolve by the Kaup-Kuperschmidt equation:

$$u_t = u''''' - 5uu''' - \tfrac{25}{2}u'u'' + 5u^2 u'.$$

(see Case 3 in §3 of their paper, taking $\lambda = 0$). In fact, up to a multiplicative factor of 1/3, Chou and Qu's flow is the same as the restriction of Z_3 to conical curves.

Not only does flow Z_3 give a geometric realization of the Kaup-Kuperschmidt equation, but the entire sequence (58) of flows realize the Kaup-Kuperschmidt hierarchy, when restricted to conical curves. To see this, note that the square of the recursion operator $\tilde{\mathcal{R}}$ relates the evolution of $k_1 = u$ under Z_j to its evolution under Z_{j+4}. (Here, we use the notation of §5.3 but with $C = 0$ and $v = 0$ because of conicity.) The resulting recursion operator is

$$-\tfrac{1}{3}D^6 + 2uD^5 + 6u'D^3 + \left(\tfrac{49}{6}u'' - 3u^2\right)D^2 + \left(\tfrac{35}{6}u''' - 10uu''\right)D + \tfrac{13}{6}u'''' - \tfrac{41}{6}uu'' - \tfrac{23}{4}(u')^2 + \tfrac{4}{3}u^3 \\ + u'D^{-1} \circ \left(\tfrac{1}{3}u^2 - \tfrac{1}{6}u''\right) + \tfrac{1}{3}\left(u''''' - 5uu''' - \tfrac{5}{6}u'u'' + 5u^2 u'\right)D^{-1}.$$

This agrees with the known recursion operator for symmetries of the Kaup-Kuperschmidt hierarchy. (See, e.g., Example 2.20 in [26], where the operator differs by changing u to $-u$ and rescaling time by a factor of 1/3.) Using this, one can check that the curvature flows induced by (58) for conical curves coincide with the commuting flows of the Kaup-Kuperschmidt hierarchy.

Remark 5.11. The curve flow discovered by Chou and Qu is in fact also defined for centroaffine curves parametrized proportional to projective arclength (i.e., those for which $k_2 + \frac{1}{2}k_1'$ is a constant), and nevertheless still induces Kaup-Kuperschmidt evolution for k_1. Recently, Musso [18] has extended this to a hierarchy of flows for arclength-parametrized curves in $\mathbb{RP}^2$ which induce the Kaup-Kuperschmidt hierarchy for curvature evolution. We suspect that these flows coincide with the restrictions of the flows studied in §5.3 (for $C \neq 0$) to the centroaffine lifts of such curves in $\mathbb{RP}^2$.

Remark 5.12. Schwartz and Tabachnikov [25] showed that certain maps defined on the space of convex polygons preserve the subset of polygons that are inscribed (or circumscribed) on a conic: that is, if the vertices of the polygon (or those of its projective dual) lie on a conic, then the same is true for its image under the map. The building blocks for these maps are elementary maps T_r that associate to a given polygon another polygon obtained from the intersections of diagonals joining each vertex to the vertex located r positions to left or right of it. In fact, the maps preserving conicity are precise combinations of T_r for certain values of r.

The map corresponding to $r = 2$ is called the *pentagram map* and it is known to be a discretization (in both time and space) of the Boussinesq equation [21]. It is natural to wonder if the maps in [25] are somehow associated to flows in the Boussinesq hierarchy. We are currently investigating this.

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References

[1] V.I. Arnold, B.A. Khesin, Topological methods in hydrodynamics, Springer, 1999.
[2] J-L. Brylinski, Loop spaces, characteristic classes, and geometric quantization Birkhäuser, 2007.
[3] A. Calini, T. Ivey, G. Marí Beffa, *Remarks on KdV-type Flows on Star-Shaped Curves*, Physica D **238** (2009), 788–797.
[4] K-S. Chou, C. Qu, *Integrable motions of space curves in affine geometry*, Chaos, Solitions, Fractals **14** #1 (2002), 29–44.
[5] A. Doliwa, P.M. Santini, *An elementary geometric characterization of the integrable motion of a curve*, Phys. Lett. A **185** #4 (1994), 373–384.
[6] V.G. Drinfel'd, V.V. Sokolov, *Lie algebras and equations of Korteweg– de Vries type,* pages 81–180 in Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.
[7] R. Dickson, F Gesztesy, and K. Unterkofler, *Algebro-Geometric Solutions of the Boussinesq Hierarchy*, Rev. Math. Phys. **11** (1999), 823–879.
[8] R. Hasimoto, *A soliton on a vortex filament*, J. Fluid Mech. **51** (1972), 477–485.
[9] R. Huang, D.A. Singer, *A new flow on starlike curves in* $\mathbb{R}^3$, Proc. A.M.S. **130** (2002), 2725–2735.
[10] G.L. Lamb, *Solitons on moving space curves*, J. Math. Phys. **18** (1977) 1654–1661.
[11] J. Langer, *Recursion in curve geometry*, New York J. Math. **5**, (1999), 25–51.

[12] J. Langer, R. Perline, *Poisson geometry of the filament equation*, J. Nonlinear Sci. **1** #1 (1991), 71–93.
[13] ——, *Local geometric invariants of integrable evolution equations*, J. Math. Phys. **35** (1994), 1732–1737.
[14] G. Marí Beffa, *On bi-Hamiltonian flows and their realizations as curves in real homogeneous manifold*, Pacific Journal of Mathematics **247** #1 (2010), 163–188.
[15] G. Marí Beffa, *The theory of differential invariants and KdV Hamiltonian evolutions*, Bull. Soc. Math. France, **127**(3) (1999) 363–391.
[16] G. Marí Beffa, *Poisson brackets associated to the conformal geometry of curves*, Trans. Amer. Math. Soc. **357** (2005) 2799-2827.
[17] J. E. Marsden, A. Weinstein, *Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids*, Physica D **7** #1–3 (1983), 305–323.
[18] E. Musso, *Motions of Curves in the Projective Plane Inducing the Kaup-Kuperschmidt Hierarchy*, SIGMA 8 (2012), 030, 20 pages.
[19] K. Nakayama, *Motion of curves in hyperboloids in Minkowski space*, J. Phys. Soc. Japan **67** (1998), 3031–3037.
[20] P. Olver, Applications of Lie Groups to Differential Equations (2nd ed.), Springer, 1993.
[21] V. Ovsienko, R. Schwartz, S. Tabachnikov, *The Pentagram Map: A Discrete Integrable System*, Comm. Math. Phys. **299**#2 (2010), 409–446.
[22] V. Ovsienko, S. Tabachnikov, Projective differential geometry old and new, Cambridge University Press, 2005.
[23] U. Pinkall, *Hamiltonian flows on the space of star-shaped curves*, Result. Math. **27** (1995), 328–332.
[24] J. Sanders, J-P. Wang, *Integrable systems and their recursion operators*, Nonlinear Analysis **45** (2001), 5213–5240.
[25] R. Schwartz, S. Tabachnikov, *Elementary Surprises in Projective Geometry*, Math. Intell. **32** #3 (2010), 31–34.
[26] J-P. Wang, *A List of* $1+1$*-Dimensional Integrable Equations and Their Properties*, J. Nonlinear Math. Phys. 9 (2002), Supplement 1, 213–233.
[27] E.J. Wilczynski, Projective differential geometry of curves and ruled surfaces, B.G. Teubner 1906.