

# HAMILTONIAN EVOLUTIONS OF TWISTED POLYGONS IN $\mathbb{RP}^n$

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ABSTRACT. In this paper we find a discrete moving frame and their associated invariants along projective polygons in  $\mathbb{RP}^n$ , and we use them to describe invariant evolutions of projective  $N$ -gons. We then apply a reduction process to obtain a natural Hamiltonian structure on the space of projective invariants for polygons, establishing a close relationship between the projective  $N$ -gon invariant evolutions and the Hamiltonian evolutions on the invariants of the flow. We prove that *any* Hamiltonian evolution is induced on invariants by an invariant evolution of  $N$ -gons - what we call a projective realization - and both evolutions are connected explicitly in a very simple way. Finally, we provide a completely integrable evolution (the Boussinesq lattice related to the lattice  $W_3$ -algebra), its projective realization in  $\mathbb{RP}^2$  and its Hamiltonian pencil. We generalize both structures to  $n$ -dimensions and we prove that they are Poisson, defining explicitly the  $n$ -dimensional generalization of the planar evolution (a discretization of the  $W_n$ -algebra). We prove that the generalization is completely integrable, and we also give its projective realization, which turns out to be very simple.

## 1. INTRODUCTION

Studies have shown a close relationship between evolution of curves invariant under a group action on the one hand, and completely integrable systems on the other. The best known such relation was established by Hasimoto in [11] where he showed that the Vortex-Filament flow (VF) - a curve flow in Euclidean space, invariant under the Euclidean group - induces the nonlinear Schrödinger evolution (NLS) on the curvature and torsion of the curve flow. One can state this fact by describing VF as an Euclidean realization of NLS. Many such examples followed in [19, 20, 21, 23, 28, 29, 30, 33, 34] and others, showing realizations in all classical geometries - both affine and non-affine, in the space of pure Spinors, Lagrangian Grassmannians, and more. The systems realized include KdV, mKdV, Adler-Gel'fand-Dikii (AGD) flows (defined in fact by Lax), Sowada-Koterra systems, NLS, both Camassa-Holm and modified Camassa-Holm equations, and more, with one equation sometimes realized in several different geometries.

All of these systems are biHamiltonian and at the heart of many of the studies one finds natural Hamiltonian structures defined on the curvatures of the flow. One such structure was described for invariants of curves on any homogeneous manifold, and proved to be linked directly to invariant curve evolutions in [17] and [18], ensuring that *any* of its Hamiltonian evolutions had a geometric realization. The reduction of a second, compatible structure usually indicated the existence of an associated integrable system. Establishing the relation between Hamiltonian

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structures on curvatures and invariant curve flows was facilitated by the definition of group-based moving frames and the theoretical framework around this concept (see [8]).

In a recent paper ([15]) the authors developed the discrete version of group-based moving frames and used it to study induced completely integrable systems on discrete curvatures (or invariants) by invariant evolutions of polygons in different geometric settings. In particular they found projective and centro-affine discrete realizations of the modified Volterra and Toda lattices, and a realization of a Volterra-type equation in the Euclidean 2-sphere. They also associated Hamiltonian structures to these systems, although it was not at all clear how they could be obtained in general.

In this paper we study the case of twisted gons in  $\mathbb{RP}^n$  (they are twisted to ensure that invariants are periodic). The space of polygons, as related to integrable systems, has gained in relevance lately because of its connection to the pentagram map (see [27]). Indeed, the study of generalizations of the pentagram map was originally the motivation for GMB in the pursue of the discrete interpretation of a moving frame. In [27] the authors proved that the pentagram map (a very simple map taking a polygon to the polygon obtained by intersecting the segments that join every other vertex) was a completely integrable discretization of Boussinesq equation (the second AGD flow). Later work ([14]) showed that some generalizations defined in [22] discretizing higher order AGD flows were also completely integrable. Also inspired by the pentagram map, the author of [16] defined completely integrable systems on invariants of planar projective polygons, obtained by reduction from a centro-affine case. In this paper we do not work on maps but on differential-difference systems, but it is our hope that the resulting Poisson structures might also be relevant to this area.

In the first part of the paper we describe background information on discrete moving frames and Poisson-Lie groups. The paper is aimed at two different audiences, the math-physics/geometry audience and the nonlinear-sciences/integrable systems one. Accordingly we have tried to include enough background for both and we have worked out the case of  $\mathbb{RP}^2$  in detail throughout the paper. In section 3 we proceed to describe a projective discrete moving frame along projective  $N$ -gons, choosing a particular frame that will produce invariants fitting our purposes. Section 4 describes how we can find explicitly, algorithmically and without any previous knowledge of the moving frame, the evolution induced on the  $N$ -gon invariants by an invariant evolution of  $N$ -gons. The case of the projective plane already hints clearly at a direct connection to Hamiltonian systems even at this stage.

In section 5 we prove that in the discrete case there also exists a naturally defined Hamiltonian structure on the moduli space of twisted  $N$ -gons. We obtain this structure by describing the moduli space of twisted  $N$ -gons as a quotient. We then used a well-known  $R$ -matrix on  $\mathfrak{sl}(n, \mathbb{R})$  and the *twisted quotient structure* in [31] to induce a Poisson bracket on the quotient. We also conclude that this reduced bracket can be obtained by simply evaluating the Sklyanin bracket associated to a simple parabolic tensor (which is not an  $R$ -matrix) on extended gradients.

We then connect the reduced Hamiltonian evolutions directly to invariant evolutions of projective  $N$ -gons; this is proved in section 6, where we show that any evolution that is Hamiltonian with respect to the reduced Poisson bracket has a

geometric realization as invariant evolution of projective  $N$ -gons. We also establish a direct and simple relation between them, namely the gradient of the Hamiltonian needs to be equal to the vector formed by the coefficients on the discrete moving frame defining the evolution of the polygons. This last section also describes completely integrable systems in the planar and higher dimensional cases. We first show that the integrable equation induced by a well-chosen invariant evolution of  $N$ -gons is the Boussinesq lattice related to the lattice  $W_3$ -algebra under a Miura transformation. This system appeared previously in [13]. In a final unexpected twist, we prove that the group right bracket evaluated on extended gradients is a Poisson bracket in any dimension *even though it is not Poisson prior to the evaluation*. We conjecture that both brackets form a Hamiltonian pencil for the integrable discretizations of the  $W_n$ -algebras and we prove this puzzling fact for the projective plane. To end the paper we define a generalization of our planar Boussinesq lattice to any dimension  $n$  - a lattice discretization of the  $W_n$ -algebra - and we prove that it is completely integrable by explicitly constructing its local master symmetry [10]. We also describe its exceedingly simple realization as a flow of projective polygons. To conclude the paper, we describe the relation between our reduction process and the discrete Drinfel'd-Sokolov reduction in [9] and [32].

There are clear indications that a similar study should be possible in the more general setting of homogeneous manifolds of the form  $G/H$  and  $(G \times \mathbb{R}^n)/G$  with  $G$  semisimple. These cases include all well-known flat geometries (Euclidean, conformal, Grassmannian, etc). It is remarkable to us that in all the examples we have studied discretizing the geometry produces integrable systems that are discretizations of known continuous integrable systems. This follows a main principle in discrete differential geometry to not discretize equations, but rather the entire geometric context. Given that many continuous integrable systems can be realized as geometric flows, this geometric process of discretization seems to give a path to finding integrable discretizations for flows that can be realized as geometric continuous flows. Work in these and other directions is currently underway. The authors would like to express their gratitude to Professor Semenov-Tian-Shansky for e-mail exchanges and assistance.

## 2. BACKGROUND AND DEFINITIONS

This section has two different, seemingly unrelated, parts. The first part deals with discrete group-based moving frames, the second with Poisson Lie-groups, classical  $R$ -matrices and the twisted quotient Poisson bracket. The two will come together in the choice of a reduction process that will describe a Poisson structure on the moduli space of projective twisted  $N$ -gons.

**2.1. Discrete moving frames.** In this section we will describe basic definitions and facts needed along this paper on the subject of discrete group-based moving frames. They are taken from [15] and occasionally slightly modified to fit our needs.

Let  $M$  be a manifold and let  $G \times M \rightarrow M$  be the action of a group  $G$  on  $M$ . Although it is not needed, for simplicity we will assume from now on that  $G$  is a subgroup of the general linear group.

**Definition 2.1** (Twisted  $N$ -gon). *A twisted  $N$ -gon in  $M$  is a map  $\phi : \mathbb{Z} \rightarrow M$  such that for some fixed  $g \in G$  we have  $\phi(p + N) = g \cdot \phi(p)$  for all  $p \in \mathbb{Z}$ . (The notation  $\cdot$  represents the action of  $G$  on  $M$ .) The element  $g \in G$  is called *the**

*monodromy* of the polygon. We will denote a twisted  $N$ -gon by its image  $x = (x_s)$  where  $x_s = \phi(s)$ .

The main reason to work with twisted polygons is our desire to have periodic invariants. We will denote by  $P_N$  the space of twisted  $N$ -gons in  $M$ . If  $G$  acts on  $M$ , it also has a natural induced action on  $P_N$  given by the diagonal action  $g \cdot (x_s) = (g \cdot x_s)$ .

**Definition 2.2** (Discrete moving frame). Let  $G^N$  denote the Cartesian product of  $N$  copies of the group  $G$ . Elements of  $G^N$  will be denoted by  $(g_s)$ . Allow  $G$  to act on the right on  $G^N$  using the inverse diagonal action  $g \cdot (g_s) = (g_s g^{-1})$  (resp. left, using the diagonal action  $g \cdot (g_s) = (g g_s)$ ). We say a map

$$\rho : P_N \rightarrow G^N$$

is a right (resp. left) *discrete moving frame* if  $\rho$  is equivariant with respect to the action of  $G$  on  $P_N$  and the right inverse (resp. left) diagonal action of  $G$  on  $G^N$ . Since  $\rho(x) \in G^N$ , we will denote by  $\rho_s$  its  $s$ th component; that is  $\rho = (\rho_s)$ , where  $\rho_s(x) \in G$  for all  $s$ ,  $x = (x_s)$ . Clearly, if  $\rho = (\rho_s)$  is a right moving frame, then  $\rho^{-1} = (\rho_s^{-1})$  is a left moving frame, and vice versa.

In short,  $\rho$  assigns an element of the group to each vertex of the polygon in an equivariant fashion. The concept of a group-based moving frame might seem foreign to some, but it is a classical idea in the continuous case: if  $\gamma(t)$  is a curve in the Euclidean plane, we can translate it to take  $\gamma(t)$  to the origin, and rotate it so that its graph is tangent to the  $t$ -axis at  $\gamma(t)$ . The curvature at  $t$  appears in the second order term of the Taylor expansion of the transformed curve. The element of the Euclidean group performing the translation and rotation defines a map from the jet space of the curve to  $G$ ; it is indeed a right moving frame and our definition above is a discrete analogue. As in this continuous example, moving frames are determined by the normalization imposed through a choice of local section transverse to the action of the group (in the continuous Euclidean curve example the section is transverse to the orbit of the Euclidean group on the jet space of the curve and it is defined by  $\gamma = (0, 0), \gamma' = (1, 0)$ ). This is also true for its discrete version.

**Proposition 2.3.** ([15]) *Let  $\mathcal{C}$  be a collection  $\mathcal{C}_1, \dots, \mathcal{C}_N$  of local cross-sections to the orbit of  $G$  through  $x_1, \dots, x_N$ . Let  $\rho = (\rho_s) \in G^N$  be uniquely determined by the condition*

$$(1) \quad \rho_s \cdot (x_r) \in \mathcal{C}_s,$$

for any  $s$ . Then  $\rho = (\rho_s((x_r))) \in G^N$  is a right moving frame along the  $N$ -gon  $(x_r)$ .

These group elements carry the invariant information of the polygon, as we see next.

**Definition 2.4** (Discrete invariant). Let  $I : P_N \rightarrow \mathbb{R}$  be a function defined on  $N$ -gons. We say that  $I$  is a scalar *discrete invariant* if

$$(2) \quad I((g \cdot x_s)) = I((x_s))$$

for any  $g \in G$  and any  $x = (x_s) \in P_N$ .

We will naturally refer to vector discrete invariants when considering vectors whose components are discrete scalar invariants.

**Definition 2.5** (Maurer–Cartan matrix). Let  $\rho$  be a right (resp. left) discrete moving frame evaluated along a twisted  $N$ -gon. The element of the group

$$K_s = \rho_{s+1}\rho_s^{-1} \quad (\text{resp. } \rho_s^{-1}\rho_{s+1})$$

is called the right (resp. left) discrete  $s$ -Maurer–Cartan matrix for  $\rho$ . We will call the equation  $\rho_{s+1} = K_s\rho_s$  the right discrete  $s$ -Serret–Frenet equation. The element  $K = (K_s) \in G^N$  is called the right (resp. left) discrete Maurer–Cartan matrix for  $\rho$ .

One can directly check that if  $K = (K_s)$  is a discrete Maurer–Cartan matrix for the right frame  $\rho$ , then  $(K_s^{-1})$  is a left one for the left frame  $\rho^{-1} = (\rho_s^{-1})$ , and vice versa. The entries of a Maurer–Cartan matrix are functional generators of all discrete invariants of polygons, as it was shown in [15]. From now on we will assume, for simplicity, that  $M = G/H$  is homogeneous and that  $G$  acts on  $M$  via left multiplication on representatives of the class. Let us denote by  $o \in M$  the class of  $H$ .

Given a discrete right moving frame  $\rho$ , and assuming for simplicity that  $\rho_s \cdot x_s = o$  for all  $s$ , one can describe the most general formula for an invariant evolution of polygons of the form

$$(3) \quad (x_s)_t = F_s(x)$$

in terms of the moving frame. This is reflected in the following theorem, which can be found in [15]. Denote by  $\Phi_g : G/H \rightarrow G/H$  the map defined by the action of  $g \in G$  on  $G/H$ , that is  $\Phi_g(x) = g \cdot x$ .

**Theorem 2.6.** *Let  $\rho$  be a right moving frame with  $\rho_s \cdot x_s = o$  for all  $s$ . Any  $G$ -invariant evolution of the form (3) can be written as*

$$(4) \quad (x_s)_t = d\Phi_{\rho_s^{-1}}(o)(\mathbf{v}_s)$$

where  $o = [H]$ , and  $\mathbf{v}_s(x) \in T_{x_s}M$  is an invariant vector.

If a family of polygons  $x(t)$  is evolving according to (4), there is a simple process to describe the evolution induced on the Maurer–Cartan matrices, and hence on a generating set of invariants. It is described in the following theorem, which can also be found in [15], slightly modified.

**Theorem 2.7.** *Assume  $x(t)$  is a flow of polygons solution of (4) and let  $\varsigma : G/H \rightarrow G$  be a section of  $G/H$  such that  $\varsigma(o) = e \in G$ . Let  $\rho$  be a right moving frame with  $\rho_s \cdot x_s = o$  so that  $\rho_s = \rho_s^H \varsigma(x_s)^{-1}$ , for some  $\rho_s^H \in H$ . Then*

$$(5) \quad (K_s)_t = N_{s+1}K_s - K_sN_s$$

where  $K_s$  is the right Maurer–Cartan matrix and  $N_s = (\rho_s)_t \rho_s^{-1} \in \mathfrak{g}$ . Furthermore, assume  $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ , where  $\mathfrak{g}$  is the algebra of  $G$ ,  $\mathfrak{h}$  is the algebra of  $H$  and  $\mathfrak{m}$  is a linear complement (which can be identified with the tangent to the image of the section  $\varsigma$ ). Then, if  $N_s = N_s^{\mathfrak{h}} + N_s^{\mathfrak{m}}$  splits accordingly,

$$(6) \quad N_s^{\mathfrak{m}} = -d\varsigma(o)\mathbf{v}_s.$$

As we will see in our next section, in the projective case, and in most other examples, equation (5) and condition (6) completely determine the element  $N = (N_s)$  and the evolution of  $K$ , *even if we do not know the moving frame explicitly*. Notice also that (5) is very similar to equations used in Lax representations of completely integrable systems - a key point when one tries to link integrable systems to invariant evolutions of polygons, with Serret–Frenet equations and  $t$ -evolutions providing  $\lambda$ -independent AKNS representations. Examples of how to effectively apply these theorems to compute induced evolutions on invariants can be found in [15], but we will also use them in our running example - the projective plane - in our next chapter.

**2.2. Poisson-Lie groups and the twisted quotient structure.** In this section we will assume that  $G$  is semisimple and that there exists a nondegenerate invariant inner product  $\langle \cdot, \cdot \rangle$  in  $\mathfrak{g}$  that allows us to identify  $\mathfrak{g}$  and  $\mathfrak{g}^*$ . In the case of  $G = \mathrm{SL}(n+1)$  the inner product is given by the trace of the product of matrices (with a factor of  $1/2$  in the diagonal entries) so that  $E_{i,j}^* = E_{j,i}$ , if  $i \neq j$  and  $(E_{i,i} - E_{n+1,n+1})^* = E_{i,i} - E_{n+1,n+1}$ . The matrix  $E_{i,j}$  has zeroes everywhere except for the  $(i, j)$  entry, where it has a 1. The following definitions and descriptions are taken from [31] and [9], but they were given originally by Drinfel'd in [6].

**Definition 2.8** (Poisson-Lie group). A Poisson-Lie group is a Lie group equipped with a Poisson bracket such that the multiplication map  $G \times G \rightarrow G$  is a Poisson map, where we consider the manifold  $G \times G$  with the product Poisson bracket.

**Definition 2.9** (Lie bialgebra). Let  $\mathfrak{g}$  be a Lie algebra such that  $\mathfrak{g}^*$  also has a Lie algebra structure given by a bracket  $[\cdot, \cdot]_*$ . Let  $\delta : \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{g}$  be the dual map to the dual Lie bracket, that is

$$\langle \delta(v), (\xi \wedge \eta) \rangle = \langle [\xi, \eta]_*, v \rangle$$

for all  $\xi, \eta \in \mathfrak{g}^*$ ,  $v \in \mathfrak{g}$ . Assume that  $\delta$  is a one-cocycle, that is

$$\delta([v, w]) = [v \otimes \mathbf{1} + \mathbf{1} \otimes v, \delta(w)] - [w \otimes \mathbf{1} + \mathbf{1} \otimes w, \delta(v)]$$

for all  $v, w \in \mathfrak{g}$ . Then  $(\mathfrak{g}, \mathfrak{g}^*)$  is called a *Lie bialgebra*.

If  $G$  is a Lie-Poisson group, the linearization of the Poisson bracket at the identity defines a Lie bracket in  $\mathfrak{g}^*$  and  $\delta$  is called the *cobracket*. The inverse result (any Lie bialgebra corresponds to a Lie-Poisson group) is also true for connected and simply connected Lie groups, a result due to Drinfel'd ([6]).

The following definitions will be used to prove our reduction theorem.

**Definition 2.10** (Admissible subgroup). Let  $M$  be a Poisson manifold,  $G$  a Poisson-Lie group and  $G \times M \rightarrow M$  a Poisson action. A subgroup  $H \subset G$  is called *admissible* if the space  $C^\infty(M)^H$  of  $H$ -invariant functions on  $M$  is a Poisson subalgebra of  $C^\infty(M)$ .

The following proposition describes admissible subgroups.

**Proposition 2.11.** ([31]) *Let  $(\mathfrak{g}, \mathfrak{g}^*)$  be the tangent Lie bialgebra of a Poisson Lie group  $G$ . A Lie subgroup  $H \subset G$  with Lie algebra  $\mathfrak{h} \subset \mathfrak{g}$  is admissible if  $\mathfrak{h}^0 \subset \mathfrak{g}^*$  is a Lie subalgebra, where  $\mathfrak{h}^0$  is the annihilator of  $\mathfrak{h}$ .*

We will now describe the Poisson brackets that will be central to our study.

**Definition 2.12** (Factorizable Lie bialgebras and  $R$ -matrices). A Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$  is called *factorizable* if the following two conditions hold:

- (a)  $\mathfrak{g}$  is equipped with an invariant bilinear form  $\langle \cdot, \cdot \rangle$  so that  $\mathfrak{g}^*$  can be identified with  $\mathfrak{g}$  via  $\xi \in \mathfrak{g}^* \rightarrow v_\xi \in \mathfrak{g}$  with  $\xi(\cdot) = \langle v_\xi, \cdot \rangle$ ;
- (b) the Lie bracket on  $\mathfrak{g}^* \cong \mathfrak{g}$  is given by

$$(7) \quad [\xi, \eta]_* = \frac{1}{2} ([R(\xi), \eta] + [\xi, R(\eta)]),$$

where  $R \in \text{End}(\mathfrak{g})$  is a skew-symmetric operator satisfying the *modified classical Yang-Baxter equation*

$$[R(\xi), R(\eta)] = R([R(\xi), \eta] + [\xi, R(\eta)]) - [\xi, \eta].$$

$R$  is called a *classical  $R$ -matrix*. Let  $r$  be the 2-tensor image of  $R$  under the identification  $\mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \text{End}(\mathfrak{g})$ . The tensor  $r$  is often referred to as the  $R$ -matrix also.

The simplest example of  $R$ -matrix is as follows: assume that  $\mathfrak{g}$  has a gradation of the form  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$ , where  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  are dual of each other and where  $\mathfrak{g}_0$  is commutative (for example, when  $\mathfrak{g}_0$  is the Cartan subalgebra). Then it is well-known that the map  $R : \mathfrak{g} \rightarrow \mathfrak{g}$

$$(8) \quad R(\xi_+ + \xi_0 + \xi_-) = \xi_+ - \xi_-$$

defines a classical  $R$ -matrix.

Given a Poisson Lie group and its associated Lie bialgebra, we can define a similar structure on  $G^N$  (this was explained in [31]). Indeed, we equipped  $\mathfrak{g}^N = \bigoplus_N \mathfrak{g}$  with

a nondegenerate inner product given by

$$\langle X, Y \rangle = \sum_{k=1}^N \langle X_k, Y_k \rangle$$

and we extend  $R \in \text{End}(\mathfrak{g})$  to  $R \in \text{End}(\mathfrak{g}^N)$  using  $R((X_s)) = (R(X_s))$ . Then  $G^N$  is a Poisson Lie-group (with the product Poisson structure) and  $(\mathfrak{g}^N, \mathfrak{g}_R^N)$  is its tangent Lie bialgebra, where  $\mathfrak{g}_R$  denotes  $\mathfrak{g}$  with Lie bracket (7). Remark that we are abusing the notation, using  $\langle \cdot, \cdot \rangle$  and  $R$  to denote both the inner product and the  $R$ -matrix in  $\mathfrak{g}$  and  $\mathfrak{g}^N$ . We will point out the difference only when it is not clear by the context.

The Poisson structure in  $G^N$  is called the *Sklyanin bracket*, but that is not the bracket we are interested in to start with, although we will come back to it later. Indeed, given a factorizable Lie bialgebra, the author of [31] defined what is called a *twisted Poisson structure on  $G^N$* . In what follows we will give the definition of this structure and we refer the reader to [31] for explanations on how to obtain it, and to [9] (Theorem 1) for the explicit formula. (We will also show that a reduction of the twisted quotient Poisson structure with a certain choice of  $R$ -matrix is equal to the evaluation of the Sklyanin bracket on extensions with a different and simpler tensor - which is not an  $R$ -matrix; but this is not clear a-priori.)

**Definition 2.13** (Left and right gradients). Let  $\mathcal{F} : G^N \rightarrow \mathbb{R}$  be a differentiable function. We define the *left gradient* of  $\mathcal{F}$  at  $L = (L_s) \in G^N$  as the element of  $\mathfrak{g}^N$  denoted by  $\nabla \mathcal{F}(L) = (\nabla_s \mathcal{F}(L))$  satisfying

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \mathcal{F}((\exp(\epsilon \xi_s) L_s)) = \langle \nabla_s \mathcal{F}(L), \xi_s \rangle$$

for all  $s$  and any  $\xi = (\xi_s) \in \mathfrak{g}^N$ .

Analogously, we define the *right gradient* of  $\mathcal{F}$  at  $L$  as the element of  $\mathfrak{g}^N$  denoted by  $\nabla' \mathcal{F}(L) = (\nabla'_s \mathcal{F}(L))$  satisfying

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \mathcal{F}((L_s \exp(\epsilon \xi_s))) = \langle \nabla'_s \mathcal{F}(L), \xi_s \rangle$$

for all  $s$  and any  $\xi = (\xi_s) \in \mathfrak{g}^N$ .

There is a clear relation between left and right gradients. Since

$$\mathcal{F}((L_s \exp(\epsilon \xi_s))) = \mathcal{F}((\exp(\epsilon L_s \xi_s L_s^{-1}) L_s)),$$

one sees that  $\langle \nabla'_s \mathcal{F}(L), \xi_s \rangle = \langle \nabla_s \mathcal{F}(L), L_s \xi_s L_s^{-1} \rangle$ , and from here we get

$$(9) \quad \nabla'_s \mathcal{F}(L) = L_s^{-1} \nabla_s \mathcal{F}(L) L_s.$$

The twisted Poisson structure can be explicitly described as follows. Let  $\mathcal{F}, \mathcal{G} : G^N \rightarrow \mathbb{R}$  be two Hamiltonian functions. Let  $\mathcal{T}$  be the shift operator  $\mathcal{T}(X_s) = X_{s+1}$ . We define the  $\mathcal{T}$ -twisted Poisson bracket as

$$(10) \quad \begin{aligned} \{\mathcal{F}, \mathcal{G}\}(L) &= \sum_{s=1}^N r(\nabla_s \mathcal{F} \wedge \nabla_s \mathcal{G}) + \sum_{s=1}^N r(\nabla'_s \mathcal{F} \wedge \nabla'_s \mathcal{G}) \\ &- \sum_{s=1}^N (\mathcal{T} \otimes \text{id})(r)(\nabla'_s \mathcal{F} \otimes \nabla_s \mathcal{G}) + \sum_{s=1}^N (\mathcal{T} \otimes \text{id})(r)(\nabla'_s \mathcal{G} \otimes \nabla_s \mathcal{F}). \end{aligned}$$

The authors of [9, 31] proved that, not only is this a Poisson bracket, but the gauge action of  $G^N$  in itself, that is, the action  $G^N \times G^N \rightarrow G^N$

$$(L_s) \rightarrow (g_{s+1} L_s g_s^{-1}),$$

is a Poisson map and the gauge orbits are Poisson submanifolds. In section 5 we will prove that this bracket can be reduced to the space of projective invariants of twisted  $N$ -gons to produce a Hamiltonian structure which is naturally linked to projective invariant evolutions of twisted gons. Furthermore, we will prove that a reduction of (10) with a certain choice of  $R$ -matrix  $r$  coincides with the evaluation of the Sklyanin bracket

$$(11) \quad \{\mathcal{F}, \mathcal{G}\}_S(L) = \sum_{s=1}^N \hat{r}(\nabla_s \mathcal{F} \wedge \nabla_s \mathcal{G}) - \sum_{s=1}^N \hat{r}(\nabla'_s \mathcal{F} \wedge \nabla'_s \mathcal{G})$$

along functional extensions, for a different, simpler, choice of tensor  $\hat{r}$ . The tensor  $\hat{r}$  will not be an  $R$ -matrix and hence (11) will not be Poisson, but nevertheless its evaluation will be.

Finally, consider the right bracket given by

$$(12) \quad \{\mathcal{F}, \mathcal{G}\}'(L) = \sum_{s=1}^N \hat{r}(\nabla'_s \mathcal{F} \wedge \nabla'_s \mathcal{G}).$$

This bracket is not a Poisson bracket even when  $\hat{r}$  is an  $R$ -matrix. Still, we will also show that its evaluation along extensions defines a second Poisson bracket, compatible (at least in the planar case) with the previous reduction.



## 3. PROJECTIVE MOVING FRAMES AND THEIR MAURER–CARTAN MATRICES

The readers familiar with the definition of Wilczynski's projective invariants of curves ([36]) will recognize the process below as its discrete analogue. They are also described in [27].

First of all we will describe the gradation of  $\mathfrak{sl}(n+1)$  that defines  $\mathbb{RP}^n$  as parabolic manifold. We can write  $\mathfrak{sl}(n+1) = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ , where  $\xi_i \in \mathfrak{g}_i$ ,  $i = 1, 0, -1$  are of the form

$$(13) \quad \xi_1 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ * & \dots & * & 0 \end{pmatrix}, \xi_0 = \begin{pmatrix} * & \dots & * & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & \dots & * & 0 \\ 0 & \dots & 0 & * \end{pmatrix}, \xi_{-1} = \begin{pmatrix} 0 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & * \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

In what follows we will consider  $\mathbb{RP}^n$  as the homogeneous space  $\mathrm{PSL}(n+1)/H$ , where  $\mathrm{PSL}(n+1) = \mathrm{SL}(n+1)/\pm I$  is the projective linear simple group and  $H$  is the subgroup corresponding to the Lie subalgebra  $\mathfrak{g}_1 \oplus \mathfrak{g}_0$ . We will also consider the local section of the quotient  $\varsigma : U \subset \mathbb{RP}^n \rightarrow \mathrm{PSL}(n+1)$  given by

$$(14) \quad \varsigma(x) = \begin{pmatrix} I_n & x \\ 0 & 1 \end{pmatrix}$$

where  $x$  are the homogeneous affine coordinates in  $\mathbb{RP}^n$  associated to the lift  $x \in \mathbb{RP}^n \rightarrow (x, 1) \in \mathbb{R}^{n+1}$ , and where  $I_n$  is the  $n \times n$  identity matrix.

Assume  $x = (x_s)$  is a twisted  $N$ -gon in  $\mathbb{RP}^n$  and consider the lift to  $\mathbb{R}^{n+1}$  given by

$$(15) \quad V_s = t_s \begin{pmatrix} x_s \\ 1 \end{pmatrix},$$

for some  $t_s \neq 0$ . We will say a polygon is *nondegenerate* if  $V_s, \dots, V_{s+n}$  are independent for all  $s$ . The following is known, and used (without proof) for example in [14]; but we could not find a simple reference so we include a short proof.

**Lemma 3.1.** *Consider the  $N \times N$  matrix*

$$S = \begin{pmatrix} 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \dots & 0 & \dots & 0 & 1 & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 & 1 \end{pmatrix}$$

where we have a total of  $k$  consecutive 1's in each row. Then  $S$  is invertible if and only if  $N$  and  $k$  are coprimes.

*Proof.* Let  $E$  be the matrix

$$E = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Once can easily see that

$$(16) \quad S = I + E + E^2 + \dots + E^{k-1}.$$

Furthermore, the characteristic polynomial for  $E$  is  $\lambda^N - 1$  so that its eigenvalues are  $\omega^p$ ,  $p = 0, \dots, N-1$ , where  $\omega = e^{\frac{2\pi i}{N}}$  is an  $N$ th root of unity. Thus, both  $E$  and  $S$  can be diagonalized, showing that the eigenvalues of  $S$  are given by

$$\alpha_p = 1 + \omega^p + \omega^{2p} + \dots + \omega^{p(k-1)}$$

$p = 0, 1, \dots, N-1$ . Assume that  $\alpha_p = 0$  for some  $0 < p < N$ . Then  $\omega^p$  is also solution of  $\lambda^k - 1 = 0$  since  $\alpha_p(\omega^p - 1) = \omega^{pk} - 1 = 0$ . Therefore,  $\frac{p}{N} = \frac{q}{k}$  for some integer  $q = 0, 1, \dots, k-1$ . This occurs if, and only if  $N$  and  $k$  are not coprime.  $\square$

The following proposition was proved in [27] for the case  $n = 2$ , and it is stated (without proof) in [14]. From now on we will assume that  $(x_s)$  is nondegenerate.

**Proposition 3.2.** *If  $N$  and  $n+1$  are coprime, then  $t_s$  can be found so that*

$$(17) \quad \det(V_{s+n}, \dots, V_{s+1}, V_s) = 1$$

for any  $s$ .

*Proof.* Substituting (15) in (17) results in a system of equations of the form

$$t_{s+n} t_{s+n-1} \dots t_s = f_s$$

for  $s = 1, \dots, N$ , where

$$f_s^{-1} = \det(x_{s+n} - x_{s+n-1}, x_{s+n-1} - x_{s+n-2}, \dots, x_{s+1} - x_s).$$

Assume  $f_s > 0$  for all  $s$ . We need to solve for  $t_i$  using this system of equations. Applying logarithms changes the system into

$$T_{s+n} + T_{s+n-1} + \dots + T_s = F_s$$

where  $T_i = \log t_i$  and  $F_i = \log f_i$ . To prove the proposition we need to prove that the matrix of coefficients of this linear system is invertible. The coefficient matrix is the one displayed in lemma 3.1 for  $k = n+1$ , ending the proof. If  $f_s < 0$  for some  $s$ , we would adjust the sign of the  $t_s$  to obtain negative values as solution.  $\square$

Once we have defined the appropriate lifts, the definition of the left projective moving frame is rather simple. Indeed, the map  $(x_s) \rightarrow (\tilde{\rho}_s^{-1})$ , where

$$(18) \quad \tilde{\rho}_s^{-1} = (V_{s+n}, \dots, V_{s+1}, V_s) \in \mathrm{SL}(n+1)$$

defines a *left* discrete moving frame for  $x = (x_s)$  since it is equivariant (we are using the inverse notation since we reserve  $\rho$  for right moving frames). Furthermore, it is straightforward to see that  $\tilde{\rho}_s^{-1} \cdot o = x_s$  since the lift of  $o \in G/H$  is the vector  $e_{n+1} \in \mathbb{R}^{n+1}$  and  $\tilde{\rho}_s^{-1} e_{n+1} = V_s$  whose projectivization is  $x_s$ .

Once we have the left moving frame we can write down its Maurer–Cartan matrix. Indeed, since  $\{V_{s+n}, \dots, V_{s+1}, V_s\}$  generates  $\mathbb{R}^{n+1}$ , we have

$$V_{s+n+1} = \hat{k}_s^n V_{s+n} + \dots + \hat{k}_s^1 V_{s+1} + (-1)^n V_s$$

where the coefficient of  $V_s$  is determined by the condition  $\det(V_{s+n}, \dots, V_s) = 1$  for all  $s$  (and in particular for  $s+1$ ). From here, the *left* Serret–Frenet equation is

given by

$$(19) \quad \tilde{\rho}_{s+1}^{-1} = \tilde{\rho}_s^{-1} \begin{pmatrix} \hat{k}_s^n & 1 & 0 & \dots & 0 \\ \hat{k}_s^{n-1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \hat{k}_s^1 & 0 & \dots & 0 & 1 \\ (-1)^n & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The corresponding *right* Maurer–Cartan matrix (generated by  $\tilde{\rho}_s$ ) is given by the inverse of the matrix above, namely by

$$\tilde{K}_s = \tilde{\rho}_{s+1} \tilde{\rho}_s^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 1 & 0 & \dots & 0 & (-1)^{n-1} \hat{k}_s^n \\ 0 & 1 & \dots & 0 & (-1)^{n-1} \hat{k}_s^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & (-1)^{n-1} \hat{k}_s^1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 1 & 0 & \dots & 0 & \tilde{k}_s^n \\ 0 & 1 & \dots & 0 & \tilde{k}_s^{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \tilde{k}_s^1 \end{pmatrix},$$

with  $\tilde{k}_s^i = (-1)^{n-1} \hat{k}_s^i$ . For reasons that will become clear later on, this is not yet our choice of invariants, moving frame or Maurer–Cartan matrix. The final choice is described in our next proposition.

**Proposition 3.3.** *There exists an invariant map  $h : \mathbb{RP}(n+1)^N \rightarrow H^N$ , (i.e.,  $h(x) = (h_s(x)) \in H^N$  such that  $h_s(g \cdot x) = h_s(x)$  for all  $g \in \mathrm{SL}(n+1)$ ) gauging  $\tilde{K}_s$  to*

$$(20) \quad K_s = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ k_s^2 & \dots & k_s^n & 1 & k_s^1 \end{pmatrix}.$$

That is, such that  $K_s = h_{s+1} \tilde{K}_s h_s^{-1}$ .

Notice that this proposition implies the existence of a right discrete moving frame  $\rho$  such that  $\rho_s \cdot x_s = o$  ( $h \in H^N$ , which is the isotropy subgroup of  $o$ ), with  $K = (K_s)$  as its Maurer–Cartan matrix. Indeed, if  $(\tilde{\rho}_s)$  is the right discrete moving frame generating  $(\tilde{K}_s)$ , then  $(\rho_s) = (h_s \tilde{\rho}_s)$ , with  $h_s$  given by the proposition.

*Proof.* If we choose

$$h_s = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ (-1)^{n-1} \tilde{k}_{s-1}^n & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (-1)^{n-1} \tilde{k}_{s-1}^2 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

then

$$h_{s+1} \tilde{K}_s h_s^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 1 & 0 & \dots & 0 & 0 \\ (-1)^n \tilde{k}_{s-1}^n & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ (-1)^n \tilde{k}_{s-1}^2 & 0 & \dots & 1 & \tilde{k}_s^1 \end{pmatrix}$$

Furthermore, if a matrix is of the form

$$M_s = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & (-1)^n \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \dots & \vdots & & \\ 0 & \dots & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ 0 & \dots & 0 & a_s^n & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \dots & \dots & \vdots & \ddots & \ddots & \ddots & \vdots & \\ 0 & \dots & 0 & a_s^{k+1} & 0 & \ddots & 1 & 0 & 0 \\ a_s^2 & \dots & a_s^{k-1} & a_s^k & 0 & \dots & 0 & 1 & a_s^1 \end{pmatrix},$$

then, after gauging it by the matrix

$$h_s = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & -a_{s-1}^n & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -a_{s-1}^{k+2} & 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

the column shifts once to the right and it transforms into

$$h_{s+1}M_s h_s^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & (-1)^n \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \dots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & \ddots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & a_{s-1}^n & 1 & 0 & \ddots & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & a_{s-1}^{k+2} & \ddots & 0 & 1 & 0 & 0 \\ a_s^2 & \dots & a_s^{k-1} & a_s^k & a_{s-1}^{k+1} & \dots & 0 & 0 & 1 & a_s^1 \end{pmatrix}.$$

We can reiterate this process  $n - 2$  times to finish the proof of the theorem.  $\square$

Recall the gradation  $\mathfrak{sl}(n+1) = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$  as in (13), where the Lie subalgebra of  $H$  is  $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$  and where  $\mathfrak{g}_{-1}$  represents the tangent to the section  $\zeta(\mathbb{R}\mathbb{P}^n)$  at the identity. Then, under this gauge,  $(K_s)_t K_s^{-1} \in \mathfrak{g}_1$  for all  $s$ . This is indeed the main advantage of adopting this choice of Maurer–Cartan matrix, as it will readily allows us to relate the invariant vector  $\mathbf{v}_s$  in (4) to the variational derivative of a Hamiltonian function. (This will be explained in section 6.) To finish with this section, we will describe (4) in our new frame  $\rho$  and will start working on our

running example. Since  $\rho_s^{-1} = \tilde{\rho}_s^{-1} h_s^{-1}$  where  $h_s$  is of the form

$$h_s = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ h_{21} & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ h_{n1} & \dots & h_{nn-1} & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

the new left moving frame will be given by

$$(21) \quad \rho_s^{-1} = (V_{s+n}, \dots, V_{s+1}, V_s) h_s^{-1} = (W_{s+n}, \dots, W_{s+1}, W_s)$$

where  $W_{s+k} = V_{s+k} + \sum_{i=1}^{k-1} h^{n-i+1, n-k+1} V_{s+i}$ ,  $k \geq 1$  and  $W_s = V_s$ . Therefore, using (4), in this new frame a general invariant evolution of polygons can be described as the projectivization of the lifted evolution

$$(22) \quad (V_s)_t = (W_{s+n} \quad \dots \quad W_{s+1} \quad W_s) \begin{pmatrix} \mathbf{v}_s \\ v_s^0 \end{pmatrix}$$

for some invariant vector  $\mathbf{v}_s$  and with  $v_s^0$  chosen uniquely in terms of  $\mathbf{v}$  and the invariants so that the normalization condition (17) is preserved.

#### Example 3.4.

Our running example will be the projective plane. The  $\mathbb{RP}^1$  case was studied in [15]. In the planar case our first left moving frame is given by  $\tilde{\rho}_s^{-1} = (V_{s+2}, V_{s+1}, V_s)$  with  $V_{s+3} = \hat{a}_s V_{s+2} + \hat{b}_s V_{s+1} + V_s$ . The left Maurer–Cartan matrix is

$$\hat{K}_s = \begin{pmatrix} \hat{a}_s & 1 & 0 \\ \hat{b}_s & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the right one is its inverse

$$\tilde{K}_s = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -\hat{a}_s \\ 0 & 1 & -\hat{b}_s \end{pmatrix}.$$

(The reader should not be confused with the notation in [27], where  $\hat{a}$  and  $\hat{b}$  are represented by  $a$  and  $b$ .) We can gauge this matrix using

$$h_s = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_{s+1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -\hat{a}_s \\ 0 & 1 & -\hat{b}_s \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -\hat{a}_s + \alpha_{s+1} \\ -\alpha_s & 1 & -\hat{b}_s \end{pmatrix}$$

which, after choosing  $\alpha_s = \hat{a}_{s-1}$  becomes

$$K_s = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ a_s & 1 & b_s \end{pmatrix}$$

where  $a_s = -\hat{a}_{s-1}$  and  $b_s = -\hat{b}_s$ . The new left moving frame is given by

$$\rho_s^{-1} = \tilde{\rho}_s^{-1} h_s^{-1} = (V_{s+1} + a_s V_{s+1}, V_{s+1}, V_s)$$

and an invariant evolution of polygons in this frame will be given by the projectivization of

$$(23) \quad (V_s)_t = v_s^1(V_{s+2} + a_s V_{s+1}) + v_s^2 V_{s+1} + v_s^0 V_s$$

where  $v^1$  and  $v^2$  are arbitrary functions of  $a_r$  and  $b_r$ ,  $r = 1, \dots, N$ , and  $v^0$  is uniquely determined by the preservation of the normalization of the lift  $V_s$ .

#### 4. AN EXPLICIT FORMULA FOR THE EVOLUTION OF THE INVARIANTS

We now turn to the following question: if we have an invariant evolution of the form (4), where  $\rho_s$  is the (right) moving frame associated to  $K_s$ , how can we effectively obtain the explicit evolution of the invariants  $k_s^i$ ? A partial answer is given by theorem 2.7. If we choose the section  $\varsigma$  as in (14), we can write  $N = (N_s) \in \mathfrak{sl}(n+1)^N$ , with  $N_s = (\rho_s)_t \rho_s^{-1}$ , as

$$(24) \quad N_s = \begin{pmatrix} \Gamma_s & \mathbf{v}_s \\ \mathbf{n}_s^T & -\text{tr}(\Gamma_s) \end{pmatrix}$$

where  $\mathbf{v}_s$  is given by (4). The matrix  $N_s$  will satisfy the structure equations

$$(25) \quad (K_s)_t = N_{s+1} K_s - K_s N_s.$$

Luckily, these equations also allow us to solve for  $\Gamma_s$  and  $\mathbf{n}_s$  for all  $s$  and we can write  $N_s$  in terms of  $\mathbf{v}$  and  $\mathbf{k}$  only. Here we are using bold face for discrete vectors, with  $\mathbf{v} = (v^i)$ ,  $\mathbf{k} = (k^i)$  and  $v^i = (v_s^i)$ ,  $k^i = (k_s^i)$ .

**Theorem 4.1.** *Assume the Maurer–Cartan element  $(K_s)$  is defined by (20), and assume  $N_s$  is defined as in (24). Assume further than the operator*

$$(26) \quad \mathcal{T} + 1 + \mathcal{T}^{-1} + \dots + \mathcal{T}^{-(n-1)}$$

*is invertible. Then, the structure equations (25) determine uniquely  $N_s$  and  $(k_s^i)_t$ ,  $s = 1, \dots, N$ ,  $i = 1, \dots, n$  as functions of  $\mathbf{v}$  and  $\mathbf{k}$ .*

*Proof.* Let us write  $K_s$  and  $N_s$  as

$$\begin{pmatrix} \Lambda & (-1)^n e_1 \\ \overline{\mathbf{k}}_s^T + e_n^T & k_s^1 \end{pmatrix}, \quad N_s = \begin{pmatrix} \Gamma_s & \mathbf{v}_s \\ \mathbf{n}_s^T & -\text{tr}(\Gamma_s) \end{pmatrix},$$

where  $\overline{\mathbf{k}}_s^T = (k_s^2, k_s^3, \dots, k_s^n, 0)$  is the truncated vector,

$$\Lambda = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and where  $\Gamma_s$  and  $\mathbf{n}_s$  are still to be found. With this notation the structure equations can be written as

$$\begin{pmatrix} 0 & 0 \\ (\overline{\mathbf{k}}_s^T)_t & (k_s^1)_t \end{pmatrix} =$$

$$\begin{pmatrix} \Gamma_{s+1}\Lambda - \Lambda\Gamma_s + (-1)^{n-1}e_1\mathbf{n}_s^T & (-1)^n(\Gamma_{s+1}e_1 + \text{tr}(\Gamma_s)e_1) \\ +\mathbf{v}_{s+1}(\overline{\mathbf{k}}_s^T + e_n^T) & +k_s^1\mathbf{v}_{s+1} - \Lambda\mathbf{v}_s \\ \mathbf{n}_{s+1}^T\Lambda - k_s^1\mathbf{n}_s^T & \\ -(\overline{\mathbf{k}}_s^T + e_n^T)(\Gamma_s + \text{tr}(\Gamma_{s+1})I) & * \end{pmatrix}.$$

This equality implies conditions

$$(27) \quad \Gamma_{s+1}\Lambda - \Lambda\Gamma_s + \mathbf{v}_{s+1}(\overline{\mathbf{k}}_s^T + e_n^T) + (-1)^{n+1}e_1\mathbf{n}_s^T = 0$$

$$(28) \quad (-1)^n\Gamma_{s+1}e_1 + k_s^1\mathbf{v}_{s+1} - \Lambda\mathbf{v}_s + (-1)^n\text{tr}(\Gamma_s)e_1 = 0.$$

and it describes  $(k_s^i)_t$  in terms of  $\Gamma_s$  and  $\mathbf{n}_s$ , for all  $s$  and  $i$ . The first row of equation (27) gives us

$$e_1^T\Gamma_{s+1}\Lambda + \mathbf{v}_{s+1}^T e_1(\overline{\mathbf{k}}_s^T + e_n^T) + (-1)^{n+1}\mathbf{n}_s^T = 0$$

which will solve for  $\mathbf{n}_s$  once  $\Gamma_s$  has been found. The  $p$  column of (27),  $p \neq n$ , is given by

$$(29) \quad \Gamma_{s+1}e_{p+1} - \Lambda\Gamma_s e_p + \mathbf{v}_{s+1}k_s^{p+1} + (-1)^{n+1}e_1\mathbf{n}_s^T e_p = 0.$$

This will solve for  $\Gamma_s e_p$ , except for the last entry, in terms of  $\Gamma_{s+1}e_{p+1}$ , for all  $p \neq n$ . Next we notice that the last column of (27) is given by

$$(30) \quad -\Lambda\Gamma_s e_n + \mathbf{v}_{s+1} + (-1)^{n+1}\mathbf{n}_s^T e_n = 0,$$

which solves for all entries of  $\Gamma_s e_n$  with the exception of its last entry. Denote  $\Gamma_s = (n_{i,j}^s)$ . Using (29) and (30) we have solved for  $n_{i,n}^s$ ,  $i = 1, \dots, n-1$ , and using (29) we have solved for  $n_{i,j}$  with  $i < j$ , and we have found the recursion

$$(31) \quad n_{i,j}^s = n_{i+1,j+1}^{s+1} + \mathbf{v}_{s+1}^T e_{i+1} k_s^i$$

whenever  $i \geq j$ .

Finally, we use (28), whose entries other than the first one solve for  $n_{i,1}$ ,  $i = 2, 3, \dots, n$ . These entries and the recursion above determines  $n_{i,j}^s$  for all  $i > j$ . As a last step, the first entry of (28) gives us

$$(32) \quad (-1)^n n_{1,1}^{s+1} + k_s^1 \mathbf{v}_{s+1}^T e_1 + (-1)^n \text{tr}(\Gamma_s) = 0.$$

But according to the recursion (31),  $n_{i,i}^s = n_{1,1}^{s-i+1} + X_i^s$ , where  $X_i^s$  is an expression depending on  $\mathbf{v}$  and  $\mathbf{k}$ . Therefore

$$\text{tr}(\Gamma_s) = \sum_{i=1}^n n_{i,i}^s = \sum_{i=1}^n n_{1,1}^{s-i+1} + X_s.$$

Substituting this relation in equation (32) we finally have

$$(\mathcal{T} + 1 + \mathcal{T}^{-1} + \dots + \mathcal{T}^{-(n-1)})n_{1,1}^s = Y_s$$

where, again,  $Y_s$  is a function depending on  $\mathbf{v}$  and  $\mathbf{k}$ . Using our last hypothesis,  $\mathcal{T} + 1 + \mathcal{T}^{-1} + \dots + \mathcal{T}^{-(n-1)}$  is invertible so that we can finally solve for  $n_{1,1}^s$ , and with it all other entries of  $N_s$ .  $\square$

Hypothesis (26) is not very restrictive. In fact, the operator  $\mathcal{T} + 1 + \mathcal{T}^{-1} + \dots + \mathcal{T}^{-(n-1)}$  is invertible whenever  $N$  and  $n+1$  are coprime since, as a linear map, it is represented by the matrix  $S$  in lemma (3.1) - possibly up to some row exchanges.

Although in the general case the explicit expression of the evolution is too involved to be displayed here, working out a particular example can be done algebraically in an algorithmic fashion.

**Example 4.2.**

When  $n = 2$  the Maurer–Cartan matrix and the matrix  $N_s$  are given by

$$(33) \quad K_s = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ a_s & 1 & b_s \end{pmatrix}, \quad N_s = \begin{pmatrix} n_{11}^s & n_{12}^s & v_s^1 \\ n_{21}^s & n_{22}^s & v_s^2 \\ n_{31}^s & n_{32}^s & -n_{11}^s - n_{22}^s \end{pmatrix},$$

where  $\mathbf{v}_s = \begin{pmatrix} v_s^1 \\ v_s^2 \end{pmatrix}$  are given by the evolution (23). From now on, and for simplicity's sake, we will denote  $a_s$  merely by  $a$  and  $a_{s+p} = \mathcal{T}^p a$ , for any  $p$ . Likewise with other functions. The structure equations, written as  $(K_s)_t K_s^{-1} = N_{s+1} - K_s N_s K_s^{-1}$ , become

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (b)_t & (a)_t & 0 \end{pmatrix} = \begin{pmatrix} (\mathcal{T} + 1)n_{11} + n_{22} + bn_{32} & \mathcal{T}n_{12} - n_{31} + an_{32} & \mathcal{T}v^1 - n_{32} \\ \mathcal{T}n_{21} - v^1 + bn_{12} & \mathcal{T}n_{22} - n_{11} + an_{12} & \mathcal{T}v^2 - n_{12} \\ \mathcal{T}n_{31} - v^2 - av^1 & \mathcal{T}n_{32} - n_{21} - bn_{31} & * \\ +b(n_{11} + 2n_{22} + an_{12} + bn_{32}) & +a(n_{22} - n_{11} + an_{12} + bn_{32}) & \end{pmatrix}.$$

These equations completely determine the entries of  $N_s$  to be given by

$$\begin{aligned} n_{32} &= \mathcal{T}v^1 & n_{21} &= \mathcal{T}^{-1}v^1 - \mathcal{T}^{-1}b\mathcal{T}v^2 & n_{11} &= (\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}(\mathcal{T}^{-1}a\mathcal{T}v^2 - b\mathcal{T}v^1) \\ n_{12} &= \mathcal{T}v^2 & n_{31} &= \mathcal{T}^2v^2 + a\mathcal{T}v^1 & n_{22} &= -(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}((\mathcal{T}^{-1} + 1)a\mathcal{T}v^2 + \mathcal{T}^{-1}b\mathcal{T}v^1). \end{aligned}$$

They also determine the evolution of  $b_s$  and  $a_s$ . This evolution can be written as

$$(34) \quad \begin{pmatrix} a \\ b \end{pmatrix}_t = \mathcal{P} \begin{pmatrix} \mathcal{T}v^2 \\ \mathcal{T}v^1 \end{pmatrix}$$

where

$$(35) \quad \mathcal{P} = \begin{pmatrix} \mathcal{T}^{-1}b - b\mathcal{T} & \mathcal{T} - \mathcal{T}^{-2} \\ +a(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}a & +a(1 - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b \\ \mathcal{T}^2 - \mathcal{T}^{-1} & \mathcal{T}a - a\mathcal{T}^{-1} \\ -b(1 - \mathcal{T})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}a & +b(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b \end{pmatrix}.$$

Later, in section 5, we will show that  $\mathcal{P}$  is a Poisson tensor for any dimension  $n$ . The tensor (35) also appears in [16].

## 5. THE PROJECTIVE HAMILTONIAN STRUCTURE ON THE SPACE OF INVARIANTS

In this section we aim to describe a naturally defined Poisson structure on the space of Maurer–Cartan matrices, and to give a precise account of how to obtain such a structure explicitly. The structure will be obtained via a reduction process from the twisted quotient structure on the Poisson-Lie group  $\mathrm{SL}(n+1)^N$ . The first step in our reduction process is to describe the space of Maurer–Cartan matrices locally as a quotient of the Poisson-Lie group  $\mathrm{SL}(n+1)^N$ , and to find explicitly the gradients of functional extensions that are constant on the leaves defining the quotient.



**5.1. The space of invariants as a quotient space.** Assume we have a nondegenerate twisted polygon  $x = (x_s)$  in a manifold  $M = G/H$  with associated right moving frame  $\rho$  such that  $\rho_s \cdot x_s = o$  for all  $s$ .

The subgroup  $H^N$  acts naturally on  $G^N$  via the gauge transformation

$$(g_s) \rightarrow (h_{s+1}g_s h_s^{-1})$$

(assuming  $h_{s+N} = h_s$  for all  $s$ ) and it is natural to ask what the quotient  $G^N/H^N$  represents. The following result is valid for any homogeneous space.

**Theorem 5.1.** *Locally around a nondegenerate polygon, a section of the quotient  $G^N/H^N$  can be described by the right Maurer–Cartan matrix  $K$  associated to the right moving frame  $\rho$ . That is, let  $x \in G^N/H^N$  be a nondegenerate twisted polygon,  $\mathcal{U} \subset G^N/H^N$  an open set of  $G^N/H^N$  containing nondegenerate twisted polygons, and let  $\mathcal{K}$  be the set of all the Maurer–Cartan matrices in  $G^N$  associated to right moving frames along elements in  $\mathcal{U}$  and determined by a fixed transverse section as in Proposition 2.3. Then the map*

$$(36) \quad \mathcal{K} \rightarrow G^N/H^N, \quad (K_s) \rightarrow [(K_s)]$$

*is a section of the quotient, a local isomorphism.*

*Proof.* Let  $\mathcal{U}$  be a neighborhood of a nondegenerate polygon  $x$  in  $G^N/H^N$ , small enough to preserve the non-degeneracy, and let's fix sections as in Proposition 2.3 uniquely determining right moving frames for  $x$  so that,  $\rho_s \cdot x_s = o$ . Clearly, the map (36) is well defined and 1-to-1, so we simply need to show that it is continuous and its image is an open set.

Let  $K \in G^N$  be a Maurer–Cartan matrix, and assume  $M \in G^N$  is near  $K$ . We will show that  $M$  can be gauged to a Maurer–Cartan element  $\hat{K}$  corresponding to some polygon  $\hat{x}$  nearby  $x$ . Define the recurrence relation

$$\eta_{s+1} = M_s \eta_s$$

for some  $\eta_0$  fixed, and let  $\hat{x}_s$  be the polygon defined by the vertices  $\eta_s^{-1} \cdot o = \hat{x}_s$ . If  $M$  is nearby  $K$ , then  $\hat{x}$  will be nearby  $x$ . We can use the same transverse sections defining  $\rho$  and  $K$  to find the Maurer–Cartan matrix  $\hat{K}$  corresponding to  $\hat{x}$ . If  $M_s$  is close enough to  $K_s$  the equations can always be solved and we can find  $\hat{K}_s$  and its moving frame  $\hat{\rho}_s$  such that  $\hat{\rho}_s \cdot \hat{x}_s = o$ . Finally denote by  $h_s$  the element  $h_s = \hat{\rho}_s \eta_s^{-1}$ . Clearly  $h_s \cdot o = o$  and so  $h_s \in H$  for all  $s$ . Also,  $\hat{\rho}_s = h_s \eta_s$  and so

$$\hat{K}_s = h_{s+1} M_s h_s^{-1}$$

which implies  $[(\hat{K}_s)] = [(M_s)]$ . This concludes the proof.  $\square$

**5.2. Extensions constant on the  $H^N$ -gauge leaves.** Let  $f : \mathcal{K} \rightarrow \mathbb{R}$  be a differentiable function on  $\mathcal{K} \subset G^N$ , viewed as a section of the quotient  $G^N/H^N$ . Assume  $\mathcal{F}$  is an extension of  $f$  to  $G^N$  such that  $\mathcal{F}$  is constant on the gauge leaves of  $H^N$ . That is, assume

$$\mathcal{F}((h_{s+1}K_s h_s^{-1})) = f((K_s))$$

for any  $s$ , any  $h \in H^N$  and any  $K \in \mathcal{K}$  as in (20). In this section we aim to describe explicitly the left and right gradients of  $\mathcal{F}$  evaluated along  $\mathcal{K}$  in terms of the gradient of  $f$  and the invariants  $k_s^i$ .

**Proposition 5.2.** *Assume  $f : \mathcal{K} \rightarrow \mathbb{R}$  is a function on  $\mathcal{K}$ , seen as a section of the quotient  $\mathrm{SL}(n+1)^N/H^N$  given by (20). Assume  $\mathcal{F}$  is an extension of  $f$  to  $\mathrm{SL}(n+1)^N$ . Then, the left gradient of  $\mathcal{F}$  along  $\mathcal{K}$  is given by*

$$\nabla_s \mathcal{F}(K) = \begin{pmatrix} Q_s & \frac{\partial f}{\partial \mathbf{k}_s} \\ \mathbf{q}_s^T & -\mathrm{tr}(Q_s) \end{pmatrix}$$

where  $\frac{\partial f}{\partial \mathbf{k}_s} = \left( (-1)^n \frac{\partial f}{\partial k_s^1}, \frac{\partial f}{\partial k_s^2}, \dots, \frac{\partial f}{\partial k_s^n} \right)^T$  and where  $Q_s$  and  $\mathbf{q}_s$  depend on  $k^i$ .

*Proof.* Consider the element of  $H$

$$Z_s = \begin{pmatrix} I_n & 0 \\ \mathbf{p}_s^T & 1 \end{pmatrix} = \exp\left( \begin{pmatrix} 0 & 0 \\ \mathbf{p}_s^T & 0 \end{pmatrix} \right) \in H$$

where  $\mathbf{p}_s = (p_s^i)$ . If  $K_s$  is given as in (20), then

$$Z_s K_s = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ k_s^2 + p_s^2 & \dots & k_s^n + p_s^n & 1 & k_s^1 + (-1)^n p_s^1 \end{pmatrix}.$$

Since  $\mathcal{F}$  is an extension of  $f$ , it coincides with  $f$  along  $\mathcal{K}$  and so  $\mathcal{F}((Z_s K_s)) = f(k_s^1 + (-1)^n p_s^1, k_s^2 + p_s^2, \dots, k_s^n + p_s^n)$ . Differentiating we get

$$\sum_{s=1}^N \langle \nabla_s \mathcal{F}, \begin{pmatrix} 0 & 0 \\ \mathbf{p}_s^T & 0 \end{pmatrix} \rangle = \sum_{s=1}^N \left( (-1)^n \frac{\partial f}{\partial k_s^1} p_s^1 + \sum_{\ell=2}^n \frac{\partial f}{\partial k_s^\ell} p_s^\ell \right)$$

and this is true for any values  $p_s^i$ . The proof of the proposition follows.  $\square$

The infinitesimal description of the fact that  $\mathcal{F}$  is constant along the gauge leaves of  $H$  is obtained by differentiating the relation  $\mathcal{F}((h_{s+1} K_s h_s^{-1})) = \mathcal{F}((K_s))$  with  $h = (h_s) = (\exp(t\xi_s)) \in H^N$ . This gives

$$\langle \nabla_s \mathcal{F}(K), \xi_{s+1} \rangle - \langle \nabla'_s \mathcal{F}(K), \xi_s \rangle = 0$$

for all  $\xi \in \mathfrak{h}^N$ . Which is the same as saying

$$(37) \quad \mathcal{T}^{-1} \nabla_s \mathcal{F} - \nabla'_s \mathcal{F} \in \mathfrak{h}^o = \mathfrak{g}_1$$

along  $\mathcal{K}$ . This property will uniquely determine  $\mathbf{q}$  and  $Q$ , the remaining entries of  $\nabla_s \mathcal{F}(K)$ .

**Theorem 5.3.** *Assume the Maurer–Cartan matrix  $K$  is defined by (20), and assume  $\mathcal{F}$  is an extension of  $f : \mathcal{K} \rightarrow \mathbb{R}$  to  $\mathrm{SL}(n+1)^N$ , constant on the gauge leaves of  $H^N$ . Assume further than the operator*

$$\mathcal{T} + 1 + \mathcal{T}^{-1} + \dots + \mathcal{T}^{-(n-1)}$$

*is invertible. Then,  $\nabla \mathcal{F}(K) = (\nabla_s \mathcal{F}(K))$  is uniquely determined by (37) as a function of the gradient of  $f$  at  $\mathbf{k} = (\mathbf{k}_s)$  and  $\mathbf{k}$ .*

*Proof.* The proof of this theorem is almost identical to the proof of theorem 4.1. From (37), along  $\mathcal{K}$  we have

$$\mathcal{T} \nabla'_s \mathcal{F} - \nabla_s \mathcal{F} \in \mathfrak{h}^o = \mathfrak{g}_1.$$

Notice also that  $\frac{\partial f}{\partial \mathbf{k}_s}$  is in the  $\mathfrak{g}_{-1}$  position in  $\nabla_s \mathcal{F}(K)$ , and hence the  $\mathfrak{g}_{-1}$  entries in  $\nabla'_s \mathcal{F}(K)$  are  $\mathcal{T}^{-1} \frac{\partial f}{\partial \mathbf{k}_s}$ . Likewise,  $\mathbf{v}_s$  was the  $\mathfrak{g}_{-1}$  entry for  $N_s$ . Furthermore, since  $(K_s)_t K_s^{-1} \in \mathfrak{g}_1$ , the structure equations (25) imply

$$\mathcal{T}N_s - K_s N_s K_s^{-1} \in \mathfrak{h}^o = \mathfrak{g}_1.$$

It suffices to choose  $\mathcal{T}\nabla'_s \mathcal{F}$  in place of  $N_s$  in the proof of theorem 4.1 to obtain the proof for our current theorem for  $\mathcal{T}\nabla'_s \mathcal{F}$  (and hence for  $\nabla_s \mathcal{F}$ ).  $\square$

Here we see both the advantages of choosing a Maurer–Cartan matrix of the form (20) and the anticipated relationship between the invariants coefficients  $\mathbf{v}_s$  and the modified gradient of the Hamiltonian  $f, \frac{\partial f}{\partial \mathbf{k}_s}$ .

To illustrate the process we work out our  $\mathbb{RP}^2$  example.

**Example 5.4.**

In the planar case, recall that  $K_s$  is given by (33). If we choose

$$Z_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_s^1 & p_s^2 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_s^1 & p_s^2 & 0 \end{pmatrix} \in H$$

we see that

$$Z_s K_s = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ a_s + p_s^2 & 1 & b_s + p_s^1 \end{pmatrix}$$

and, since  $\mathcal{F}$  is an extension of  $f$ , they satisfy

$$(38) \quad \mathcal{F}(Z_s K_s) = f(a_s + p_s^2, b_s + p_s^1).$$

Let us write  $f_a^s = \frac{\partial f}{\partial a_s}$  and  $f_b^s = \frac{\partial f}{\partial b_s}$ . Relationship (38) implies

$$\langle \nabla_s \mathcal{F}(K), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_s^1 & p_s^2 & 0 \end{pmatrix} \rangle = f_a^s p_s^2 + f_b^s p_s^1$$

and so

$$(39) \quad \nabla_s \mathcal{F}(K) = \begin{pmatrix} (f_{11})_s & (f_{12})_s & f_b^s \\ (f_{21})_s & (f_{22})_s & f_a^s \\ (f_{31})_s & (f_{32})_s & -(f_{11})_s - (f_{22})_s \end{pmatrix},$$

where all  $(f_{ij})_s$  are unknown. As we saw in (9), the right gradient can be obtained through the relation  $\nabla'_s \mathcal{F}(K) = K_s^{-1} \nabla_s \mathcal{F}(K) K_s$ .

If we substitute (39) in (37) we get the following expression. (Notice that, as before, we have dropped the subindices and denote  $(f_{11})_{s+p} = \mathcal{T}^p f_{11}$ , etc.)

$$\mathcal{T}^{-1} \begin{pmatrix} f_{11} & f_{12} & f_b \\ f_{21} & f_{22} & f_a \\ f_{31} & f_{32} & -(f_{11} + f_{22}) \end{pmatrix} - \begin{pmatrix} f_{22} + a f_a & f_a & b f_a + f_{21} \\ -a(f_{11} + 2f_{22} + b f_b + a f_a) & -f_{11} - f_{22} & f_{31} - a f_{21} \\ f_{12} + a f_b & f_b & f_{11} + b f_b \end{pmatrix} \in \mathfrak{h}^o = \mathfrak{g}_1.$$

Since  $\mathfrak{g}_1$  is given as in (13), we obtain the following equations for the entries of  $\nabla\mathcal{F}$

$$\begin{aligned} \mathcal{T}^{-1}f_{12} &= f_a & \mathcal{T}^{-1}f_{11} &= f_{22} + af_a & \mathcal{T}^{-1}f_a &= f_{31} - af_{21} - b(2f_{11} + f_{22} + bf_b + af_a) \\ \mathcal{T}^{-1}f_b &= bf_a + f_{21} & -\mathcal{T}^{-1}(f_{11} + f_{22}) &= f_{11} + bf_b & \mathcal{T}^{-1}f_{21} &= f_{32} - bf_{12} - a(f_{11} + 2f_{22} + bf_b + af_a). \end{aligned}$$

Assuming that  $\mathcal{T} + 1 + \mathcal{T}^{-1}$  is invertible (which we know is true as far as  $N$  is not a multiple of 3), and denoting  $(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}$  by  $\mathcal{R}$ , these equations determine the entries of  $\nabla\mathcal{F}$  to be

$$(40) \quad \begin{aligned} f_{12} &= \mathcal{T}f_a & f_{31} &= \mathcal{T}^{-1}f_a + a\mathcal{T}^{-1}f_b \\ & & +b\mathcal{R} &((1 - \mathcal{T})af_a + (\mathcal{T}^{-1} - \mathcal{T})bf_b) \\ f_{21} &= \mathcal{T}^{-1}f_b - bf_a & f_{32} &= \mathcal{T}^{-2}f_b + (b\mathcal{T} - \mathcal{T}^{-1}b)f_a \\ & & -a\mathcal{R} &((\mathcal{T} - \mathcal{T}^{-1})af_a + (1 - \mathcal{T}^{-1})bf_b) \\ f_{11} &= \mathcal{R}(af_a - \mathcal{T}bf_b) & f_{22} &= -\mathcal{R}((1 + \mathcal{T})af_a + bf_b) \end{aligned}$$

We are now ready to move to the investigation of the Hamiltonian picture.

**5.3. Projective Hamiltonian structure on  $\mathcal{K}$ .** In this section we finally aim to prove that the twisted discrete Poisson bracket described in our initial section can be reduced to  $\mathcal{K} \subset \mathrm{SL}(n+1)^N/H^N$ , defining a natural Poisson bracket. Consider the splitting  $\mathfrak{sl}(n+1) = \mathfrak{g}_+ \oplus \mathfrak{h}_c \oplus \mathfrak{g}_-$ , where  $\mathfrak{g}_+$  are lower triangular matrices,  $\mathfrak{g}_-$  are upper triangular ones, and  $\mathfrak{h}_c$  is the Cartan subalgebra. If  $\xi = \xi_+ + \xi_c + \xi_-$  according to the gradation above, let  $R$  be the  $R$ -matrix defined by  $R(\xi) = \xi_- - \xi_+$ , so that its associated tensor  $r$  is given by

$$(41) \quad r(\xi, \eta) = \langle \xi_-, \eta_+ \rangle, \quad \langle R(\xi), \eta \rangle = r(\xi \wedge \eta) = \langle \xi_- - \xi_+, \eta \rangle.$$

**Theorem 5.5.** *The twisted Poisson structure (10) defined on  $\mathrm{SL}(n+1)^N$  with  $r$  as in (41) is locally reducible to the quotient  $\mathrm{SL}(n+1)^N/H^N$ , and the reduced bracket coincides with the evaluation of the Sklyanin bracket (11) with tensor*

$$\hat{r}(\xi, \eta) = \langle \xi_{-1}, \eta_1 \rangle$$

along extensions, where  $\xi_{-1}$  and  $\eta_1$  correspond to the original parabolic gradation (13) defining  $\mathbb{RP}^n$ .

Notice that , since  $\mathfrak{g}_0$  is not abelian,  $\hat{r}$  is not an  $R$ -matrix.

*Proof.* To prove this theorem we will apply the classical theorem in [24], reformulated according to our situation and notation. It can be stated as follows.

*Poisson reduction theorem:* Let  $M$  be a Poisson manifold and assume that  $E \subset TM$  is an integral and regular Hamiltonian subbundle (an integrable subbundle of the bundle defining the symplectic foliation). Assume  $M/E$  is a manifold. Then, if the Poisson bracket preserves  $E$  - that is, if the bracket of two functions constant on the leaves of  $E$  is constant on the leaves of  $E$  - the Poisson bracket can be reduced to the quotient  $M/E$ . The reduction is given by the formula

$$(42) \quad \{f, h\}_{M/E}([p]) = \{\mathcal{F}, \mathcal{H}\}(p)$$

where  $\mathcal{F}, \mathcal{H} : M \rightarrow \mathbb{R}$  are any extensions of  $f, h : M/E \rightarrow \mathbb{R}$  constant on the leaves of  $E$ .

In our case  $M = \mathrm{SL}(n+1, \mathbb{R})^N$  and the Poisson bracket is the twisted quotient bracket. The subbundle  $E$  is the subbundle tangent to the  $H$ -gauge leaves, which

is a Hamiltonian subbundle since the gauge leaves are Poisson submanifolds and gauging is a Poisson action ([31]). We know that  $G^N/H^N$  is defined by  $\mathcal{K}$ , a manifold, and  $E$  is regular when restricted to generic polygons. Therefore, to prove our theorem we simply need to check that  $E$  preserves the twisted Poisson bracket.

On the other hand, since  $E$  is defined by the orbits of a Poisson action for the twisted bracket, according to [31], it preserves the bracket whenever  $H$  is admissible (see definition 2.10). According to proposition 2.11, this is true whenever  $\mathfrak{h}^0$  is a Lie subalgebra of  $\mathfrak{g}^*$ , which is finally the only condition we need to check to prove the main part of the theorem.

Recall that the Lie bracket in  $\mathfrak{g}^*$  is defined by the linearization of the twisted Poisson bracket at the identity  $e \in G$ . That is

$$[d_e\phi, d_e\varphi]_* = d_e\{\phi, \varphi\} \in \mathfrak{g}^*.$$

Since  $\mathfrak{h}^0 = \mathfrak{g}_1$ , we will look for functions  $\phi_s^i$  such that  $d_e\phi_s^i$  generate  $\mathfrak{g}_1$ . Indeed, let  $L \in G^N$  be close enough to  $e \in G^N$  so that  $L = (L_s)$  can be factored as

$$L_s = \begin{pmatrix} I_n & \ell_s \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \Theta_s & \mathbf{0} \\ \mathbf{0}^T & \theta_s \end{pmatrix} \begin{pmatrix} I_n & \mathbf{0} \\ q_s^T & 1 \end{pmatrix}$$

according to the gradation of the algebra. We define  $\phi_s^i(L) = \ell_s^i$ , where  $i$  marks the  $i$ th entry of  $\ell_s$ . Instead of calculating  $d_e\phi_s^i$  we will directly calculate the left gradient at  $L$ . Notice that if  $g \in \mathrm{SL}(n+1)$ ,

$$g \begin{pmatrix} I_n & \ell \\ \mathbf{0}^T & 1 \end{pmatrix} = \begin{pmatrix} I_n & g \cdot \ell \\ \mathbf{0}^T & 1 \end{pmatrix} g_H$$

where  $g \cdot \ell$  is the projective action of  $\mathrm{PSL}(n+1)$  in  $\mathbb{R}^n$  and  $g_H \in H$ . Therefore,

$$\phi_s^i(e^{\epsilon\xi}L) = (e^{\epsilon\xi} \cdot \ell_s)^i.$$

We can now analyze each one of the factors. If  $\xi \in \mathfrak{g}_{-1}$  the action is simply a translation and so

$$\nabla\phi_s^i(L) = \begin{pmatrix} * & * \\ e_i^T & * \end{pmatrix}.$$

If  $\xi \in \mathfrak{g}_0$ , the projective action of  $e^{\epsilon\xi}$  is linear. Therefore

$$\nabla\phi_s^i(L) = \begin{pmatrix} A_s^i & * \\ e_i^T & -\ell_s^i \end{pmatrix}$$

for  $A_s^i = (\mathbf{0} \ \dots \ \mathbf{0} \ \ell_s \ \mathbf{0} \ \dots \ \mathbf{0})$ , with the nonzero column located in the  $i$ th place. If  $\xi \in \mathfrak{g}_1$ , then the infinitesimal projective action is quadratic and straightforward calculations show that

$$\nabla\phi_s^i(L) = \begin{pmatrix} A_s^i & -\ell_s^i \ell_s \\ e_i^T & -\ell_s^i \end{pmatrix}.$$

Whenever  $L = e$ , we have that  $\ell_s = \mathbf{0}$  and  $d_e\phi_s^i = E_{n+1,i}$ , generating  $\mathfrak{g}_1$ . We know calculate  $d_e\{\phi^i, \phi^j\}$  where  $\{\cdot, \cdot\}$  is the twisted bracket (10) with the  $r$ -matrix given by (41). We want to show that  $d_e\{\phi^i, \phi^j\} \in \mathfrak{h}^0$  and so we need to show that  $\frac{d}{d\epsilon}|_{\epsilon=0}\{\phi_s^i, \phi_s^j\}(e^{\epsilon\xi}) = 0$  whenever  $\xi \in \mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{g}_0$ .

Notice that  $(d_e\phi_s^i)_{-1} = 0$  and  $(\nabla\phi_s^i(L))_{-1}$  is quadratic in  $L$ . Therefore we also have  $\frac{d}{d\epsilon}|_{\epsilon=0}(\nabla\phi_s^i(e^{\epsilon\xi}))_{-1} = 0$ .

Also,  $\nabla'\phi_s^i(e^{\epsilon\xi}) = e^{-\epsilon\xi}\nabla\phi_s^i(e^{\epsilon\xi})e^{\epsilon\xi}$ , and so

$$\frac{d}{d\epsilon}|_{\epsilon=0}\nabla'\phi_s^i(e^{\epsilon\xi}) = [d_e\phi_s^i, \xi_s] + \frac{d}{d\epsilon}|_{\epsilon=0}\nabla\phi_s^i(e^{\epsilon\xi}).$$

Since  $d_e \phi_s^i \in \mathfrak{g}_1$ , whenever  $\xi \in \mathfrak{h}$  we have that  $\frac{d}{d\epsilon}|_{\epsilon=0} (\nabla' \phi_s^i(e^{\epsilon\xi}))_{-1} = 0$ . Furthermore,  $(d_e \phi_s^i)_0 = 0$  also. From here, and given that

$$\langle \nabla_+ \phi_s^i, \nabla_- \phi_s^i \rangle = \langle \nabla_1 \phi_s^i, \nabla_{-1} \phi_s^i \rangle + \langle \nabla_+^0 \phi_s^i, \nabla_-^0 \phi_s^i \rangle,$$

where  $\nabla_+^0 \phi_s^i$  is the portion of  $\nabla_+ \phi_s^i$  in  $\mathfrak{g}_0$ , and similarly with the others, we get that  $\frac{d}{d\epsilon}|_{\epsilon=0} \{\phi_s^i, \phi_s^j\}(e^{\epsilon\xi}) = 0$ , and hence  $\mathfrak{h}^0$  is a subalgebra of  $\mathfrak{g}^*$ .

Finally, using the fact that  $\mathcal{T}\nabla'\mathcal{F} - \nabla\mathcal{F} \in \mathfrak{h}^0 = \mathfrak{g}_1$  so that  $\mathcal{T}(\nabla'\mathcal{F})_- = (\nabla\mathcal{F})_-$ , the reduced Poisson bracket can be expressed as

$$\begin{aligned} \{f, h\}(\mathbf{k}) &= \frac{1}{2} (\langle (\nabla\mathcal{F})_-, (\nabla\mathcal{H})_+ \rangle - \langle (\nabla\mathcal{F})_+, (\nabla\mathcal{H})_- \rangle) \\ &+ \langle (\nabla'\mathcal{F})_-, (\nabla'\mathcal{H})_+ \rangle - \langle (\nabla'\mathcal{F})_+, (\nabla'\mathcal{H})_- \rangle - \langle \tau(\nabla'\mathcal{F})_-, (\nabla\mathcal{H})_+ \rangle + \langle \tau(\nabla'\mathcal{H})_-, (\nabla\mathcal{F})_+ \rangle \\ &= \frac{1}{2} (-\langle (\nabla\mathcal{F})_-, (\nabla\mathcal{H})_+ \rangle + \langle (\nabla\mathcal{F})_+, (\nabla\mathcal{H})_- \rangle + \langle (\nabla'\mathcal{F})_-, (\nabla'\mathcal{H})_+ \rangle - \langle (\nabla'\mathcal{F})_+, (\nabla'\mathcal{H})_- \rangle) \\ &- \frac{1}{2} (-\langle (\nabla\mathcal{F})_-, (\nabla\mathcal{H})_+ \rangle + \langle (\nabla\mathcal{F})_+, (\nabla\mathcal{H})_- \rangle + \langle (\nabla'\mathcal{F})_-, (\nabla'\mathcal{H})_+ \rangle - \langle (\nabla'\mathcal{F})_+, (\nabla'\mathcal{H})_- \rangle) \\ &= \frac{1}{2} \langle (\nabla\mathcal{H})_-, (\nabla\mathcal{F})_+ - \tau(\nabla'\mathcal{F})_+ \rangle - \frac{1}{2} \langle (\nabla\mathcal{F})_-, (\nabla\mathcal{H})_+ - \tau(\nabla'\mathcal{H})_+ \rangle. \end{aligned}$$

This is equal to

$$= \frac{1}{2} \langle (\nabla\mathcal{H})_{-1}, (\nabla\mathcal{F})_1 - \tau(\nabla'\mathcal{F})_1 \rangle - \frac{1}{2} \langle (\nabla\mathcal{F})_{-1}, (\nabla\mathcal{H})_1 - \tau(\nabla'\mathcal{H})_1 \rangle.$$

and from here we can go back to

$$- \frac{1}{2} (-\langle (\nabla\mathcal{F})_{-1}, (\nabla\mathcal{H})_1 \rangle + \langle (\nabla\mathcal{F})_1, (\nabla\mathcal{H})_{-1} \rangle + \langle (\nabla'\mathcal{F})_{-1}, (\nabla'\mathcal{H})_1 \rangle - \langle (\nabla'\mathcal{F})_1, (\nabla'\mathcal{H})_{-1} \rangle),$$

which coincides with the evaluation of (11). Even more surprising, we will later show that the right bracket (12) also produces a Poisson bracket when evaluated on extensions, even though the original bracket is also not Poisson.  $\square$

Using (42) we can actually calculate explicitly the reduction of the twisted bracket to  $\mathcal{K}$ . We will illustrate it with our running  $\mathbb{R}\mathbb{P}^2$  example.

### Example 5.6.

Assume  $f : \mathcal{K} \rightarrow \mathbb{R}$  is a Hamiltonian function and let  $\mathcal{F} : \mathrm{SL}(3)^N \rightarrow \mathbb{R}$  be an extension of  $f$ , constant on the leaves of  $H^N$ . Recall that we have explicitly found the left gradient of such a extension along  $\mathcal{K}$ , it is given by (39). We also know that the right and left gradients are conjugated and satisfy equation (37), and so the  $\mathfrak{g}_1 \oplus \mathfrak{g}_0$  component of  $\nabla'_s \mathcal{F}(K)$  equals that of  $\mathcal{T}^{-1} \nabla_s \mathcal{F}(K)$ . We have

$$\nabla \mathcal{F}(K) = \begin{pmatrix} f_{11} & f_{12} & f_b \\ f_{21} & f_{22} & f_a \\ f_{31} & f_{32} & -(f_{11} + f_{22}) \end{pmatrix}, \quad \nabla' \mathcal{F}(K) = \begin{pmatrix} \mathcal{T}^{-1} f_{11} & \mathcal{T}^{-1} f_{12} & \mathcal{T}^{-1} f_b \\ \mathcal{T}^{-1} f_{21} & \mathcal{T}^{-1} f_{22} & \mathcal{T}^{-1} f_a \\ f_{12} + a f_b & f_b & -\mathcal{T}^{-1}(f_{11} + f_{22}) \end{pmatrix},$$

where we have again dropped the subindex to avoid over-cluttering, and where the values for  $f_{ij}$  were found in (40). The reduced Poisson bracket can be obtained by simply substituting both gradients of the extensions in (10). Consider now the simplified tensor  $\hat{r}$

$$\hat{r}(\xi \otimes \eta) = \langle \xi_{-1}, \eta_1 \rangle.$$

Denote by  $g_a, g_b, g_{ij}$  the entries of  $\nabla \mathcal{G}(K)$  analogous to those of  $\nabla \mathcal{F}(K)$ . As before, we will ignore subindices, and we will write

$$\langle v, w \rangle = v^T w.$$

Substituting in (10) we obtain that the reduced bracket is given by

$$\begin{aligned}
\{f, g\}(\mathbf{k}) &= \frac{1}{2} \left\langle \begin{pmatrix} f_b \\ f_a \end{pmatrix}, \begin{pmatrix} g_{31} \\ g_{32} \end{pmatrix} \right\rangle - \frac{1}{2} \left\langle \begin{pmatrix} g_b \\ g_a \end{pmatrix}, \begin{pmatrix} f_{31} \\ f_{32} \end{pmatrix} \right\rangle \\
&+ \frac{1}{2} \left\langle \begin{pmatrix} \mathcal{T}^{-1} f_b \\ \mathcal{T}^{-1} f_a \end{pmatrix}, \begin{pmatrix} g_{12} + ag_b \\ g_b \end{pmatrix} \right\rangle - \frac{1}{2} \left\langle \begin{pmatrix} \mathcal{T}^{-1} g_b \\ \mathcal{T}^{-1} g_a \end{pmatrix}, \begin{pmatrix} f_{12} + af_b \\ f_b \end{pmatrix} \right\rangle \\
&- \left\langle \begin{pmatrix} f_b \\ f_a \end{pmatrix}, \begin{pmatrix} g_{31} \\ g_{32} \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} g_b \\ g_a \end{pmatrix}, \begin{pmatrix} f_{31} \\ f_{32} \end{pmatrix} \right\rangle \\
&= \frac{1}{2} \left\langle \begin{pmatrix} f_b \\ f_a \end{pmatrix}, \begin{pmatrix} \mathcal{T}g_{12} + \mathcal{T}ag_b - g_{31} \\ \mathcal{T}g_b - g_{32} \end{pmatrix} \right\rangle - \frac{1}{2} \left\langle \begin{pmatrix} g_b \\ g_a \end{pmatrix}, \begin{pmatrix} \mathcal{T}f_{12} + \mathcal{T}af_b - f_{31} \\ \mathcal{T}f_b - f_{32} \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} f_a \\ f_b \end{pmatrix}, \mathcal{P} \begin{pmatrix} g_a \\ g_b \end{pmatrix} \right\rangle
\end{aligned}$$

where  $\mathcal{P}$  is given as in (35).

## 6. HAMILTONIAN EVOLUTIONS OF TWISTED POLYGONS

By now it is clear that there is a very close relationship between the evolution induced on the invariants by invariant evolutions of polygons, and the Hamiltonian evolution associated to the reduced bracket obtained from the twisted bracket (10). Indeed, we have seen in our example that they are equal under some identifications. In our final section we will prove that if we choose as invariant coefficients  $\mathcal{T}\mathbf{v}_s = \frac{\partial f}{\partial \mathbf{k}_s}$ , then the evolution of the projective polygons defined by  $\mathbf{v}_s$  induces a Hamiltonian evolution on  $\mathbf{k}_s$ , with Hamiltonian function  $f$ . The result implies that *any  $n$ -dimensional reduced Hamiltonian evolution is induced on  $\mathbf{k}$  by some invariant evolution of projective polygons in  $\mathbb{RP}^n$ .*

**Theorem 6.1.** *Assume an invariant evolution of twisted  $N$ -gons in  $\mathbb{RP}^n$  lifts to an evolution of the form (22). Furthermore, assume that*

$$(43) \quad \mathcal{T}\mathbf{v}_s = \frac{\partial f}{\partial \mathbf{k}_s}$$

for some function  $f : \mathcal{K} \rightarrow \mathbb{R}$ , where  $\frac{\partial f}{\partial \mathbf{k}_s} = ((-1)^n \frac{\partial f}{\partial k_s^1}, \frac{\partial f}{\partial k_s^2}, \dots, \frac{\partial f}{\partial k_s^n})^T$ . Then, the evolution induced on the invariants  $k_s^i$  is the reduced Hamiltonian evolution associated to the Hamiltonian function  $f$ .

We will refer to (43) as the *compatibility condition*.

*Proof.* Assume  $\mathbf{k}$  evolves by an evolution that is Hamiltonian with respect to the reduced bracket on  $\mathcal{K}$ . Let's denote by  $\xi_f$  the Hamiltonian vector field associated to a Hamiltonian function  $f : \mathcal{K} \rightarrow \mathbb{R}$  such that  $\mathbf{k}_t = \xi_f(\mathbf{k})$ .

Now, given that  $\mathbf{k}_t$  appears in the  $\mathfrak{g}_1$  component of  $K_t K^{-1}$  and  $\frac{\partial f}{\partial \mathbf{k}}$  is in the  $\mathfrak{g}_{-1}$  position of  $\nabla \mathcal{F}(K)$  for any extension of  $f$  constant on the leaves of  $H$  as in (37), we have that the reduced bracket of  $f$  with any other function  $g$  can be written as

$$\{f, g\}(\mathbf{k}) = \xi_f(\mathbf{k})(g) = \langle K_t K^{-1}, \nabla \mathcal{G}(K) \rangle$$

where  $\mathcal{F}$  and  $\mathcal{G}$  are extensions of  $f, g : \mathcal{K} \rightarrow \mathbb{R}$  as in (37). Now, notice that if  $\mathcal{F}$  and  $\mathcal{G}$  satisfy (37) we have

$$(44) \quad (\nabla \mathcal{F})_{-1} = (\mathcal{T} \nabla' \mathcal{F})_{-1}, \quad \text{or} \quad (\mathcal{T}^{-1} \nabla \mathcal{F})_{-1} = (\nabla' \mathcal{F})_{-1}.$$

We are assuming that all gradients are evaluated at  $K$ . Using this relation in the Sklyanin bracket we obtain

$$\begin{aligned} \{f, g\}(\mathbf{k}) &= \langle K_t K^{-1}, \nabla \mathcal{G}(K) \rangle \\ &= \frac{1}{2} \langle (\nabla \mathcal{G})_{-1}, (\nabla \mathcal{F})_1 - \mathcal{T}(\nabla' \mathcal{F})_1 \rangle - \frac{1}{2} \langle (\nabla \mathcal{F})_{-1}, (\nabla \mathcal{G})_1 - \mathcal{T}(\nabla' \mathcal{G})_1 \rangle. \end{aligned}$$

But these expressions are skew-symmetric. Indeed, notice that  $(\nabla \mathcal{G})_1 - \mathcal{T}(\nabla' \mathcal{G})_1 = \nabla \mathcal{G} - \mathcal{T} \nabla' \mathcal{G}$  since it belongs to  $\mathfrak{g}_1$  and, therefore,

$$\langle (\nabla \mathcal{F})_{-1}, (\nabla \mathcal{G})_1 - \mathcal{T}(\nabla' \mathcal{G})_1 \rangle = \langle \nabla \mathcal{F}, \nabla \mathcal{G} - \mathcal{T} \nabla' \mathcal{G} \rangle = -\langle \nabla \mathcal{F}, \mathcal{T} \nabla' \mathcal{G} \rangle + \langle \mathcal{T} \nabla' \mathcal{F}, \mathcal{T} \nabla' \mathcal{G} \rangle$$

where we have used that  $\langle \nabla \mathcal{F}, \nabla \mathcal{G} \rangle = \langle \nabla' \mathcal{F}, \nabla' \mathcal{G} \rangle$  since  $\langle \cdot, \cdot \rangle$  is invariant under the adjoint action. From here, the above equals

$$\langle (\mathcal{T} \nabla' \mathcal{G})_{-1}, (\mathcal{T} \nabla' \mathcal{F})_1 - (\nabla \mathcal{F})_1 \rangle = -\langle (\nabla \mathcal{G})_{-1}, (\nabla \mathcal{F})_1 - (\mathcal{T} \nabla' \mathcal{F})_1 \rangle.$$

Skew-symmetry tells us that

$$\{f, g\}(\mathbf{k}) = \langle K_t K^{-1}, \nabla \mathcal{G}(K) \rangle = \langle (\nabla \mathcal{F})_1 - \mathcal{T}(\nabla' \mathcal{F})_1, (\nabla \mathcal{G})_{-1} \rangle.$$

We can now see the relation between  $\mathcal{T} \mathbf{v}$  and  $\frac{\partial f}{\partial \mathbf{k}}$ . Indeed, if  $N$  is given as in (25),

$$\langle K_t K^{-1}, \nabla \mathcal{G}(K) \rangle = \langle \mathcal{T} N - K N K^{-1}, \nabla \mathcal{G}(K) \rangle$$

for any extension  $\mathcal{G}$  as in (37). Recall next that  $\mathcal{T} N - K N K^{-1} \in \mathfrak{g}_1$ , and both  $\mathcal{T} N$  and  $\nabla \mathcal{F}$  are determined by its  $\mathfrak{g}_{-1}$  component and this condition. Notice also that  $\nabla \mathcal{F} = K \nabla' \mathcal{F} K^{-1}$ . Thus, we can conclude that

$$N = -\nabla' \mathcal{F}.$$

Since  $\mathcal{T} N_{-1} = -(\mathcal{T} \nabla' \mathcal{F})_{-1} = -(\nabla \mathcal{F})_{-1}$ , the  $\mathfrak{g}_{-1}$  component of  $N$  is  $-\mathbf{v}$ , and that of  $\nabla \mathcal{F}$  is  $\frac{\partial f}{\partial \mathbf{k}_s}$ , the theorem follows.  $\square$

**6.1. Completely integrable evolutions of planar polygons.** In this section, we study in detail the integrable lattice that appears in the case of planar polygons. Although this study and the next seem to separate themselves from invariant evolutions of polygons, we will return to connect them towards the end.

In Section 5 we have shown that the operator  $\mathcal{P}$  defined by (35) is Hamiltonian, which naturally leads to Hamiltonian evolutions for the invariants as stated in theorem 6.1. However, to obtain integrable systems we need biHamiltonian structures. To obtain one or more compatible structures, we shall introduce arbitrary constants in the operator  $\mathcal{P}$  and study the conditions on the parameters to ensure that the operator is still Hamiltonian. That is, we will analyze all possible Hamiltonian structures, compatible or not, within  $\mathcal{P}$ .

**Theorem 6.2.** *Consider the antisymmetric operator  $\mathcal{H}$  given by*

$$\left( \begin{array}{cc} \lambda_1(\mathcal{T}^{-1}b - b\mathcal{T}) & \lambda_4\mathcal{T} - \lambda_3\mathcal{T}^{-2} \\ +\lambda_2a(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}a & +\lambda_5a(1 - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b \\ \lambda_3\mathcal{T}^2 - \lambda_4\mathcal{T}^{-1} & \lambda_6(\mathcal{T}a - a\mathcal{T}^{-1}) \\ -\lambda_5b(1 - \mathcal{T})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}a & +\lambda_7b(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b \end{array} \right),$$

where  $\lambda_i$ ,  $i = 1, \dots, 7$  are constants. Then  $\mathcal{H}$  is Hamiltonian when one of the following three cases is satisfied

- (1).  $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = \lambda_7 = 0$ ,  $\lambda_3$  and  $\lambda_4$  are any constants;



- (2).  $\lambda_1\lambda_6 = \lambda_2\lambda_3$ ,  $\lambda_3 = \lambda_4$ ,  $\lambda_2 = \lambda_5 = \lambda_7$  and at least one of  $\lambda_1, \lambda_3, \lambda_6$  is nonzero;
- (3).  $\lambda_1 = \lambda_3 = \lambda_4 = \lambda_6 = 0$ ,  $\lambda_2, \lambda_5$  and  $\lambda_7$  are any constants.

*Proof.* We investigate the conditions needed for  $\mathcal{H}$  to be a Hamiltonian operator using Proposition 7.7 in [26]. Although this theorem is formulated for differential operators, it is also valid for difference operators ([25]).

For the operator  $\mathcal{H}$ , we have

$$\mathcal{H}(\xi) = \mathcal{H} \begin{pmatrix} \theta \\ \eta \end{pmatrix} = \begin{pmatrix} \lambda_1(b_{-1}\theta_{-1} - b\theta_1) + \lambda_2a(P_1 - P_{-1}) + \lambda_4\eta_1 - \lambda_3\eta_{-2} + \lambda_5a(Q - Q_{-1}) \\ \lambda_3\theta_2 - \lambda_4\theta_{-1} - \lambda_5b(P - P_1) + \lambda_6(a_1\eta_1 - a\eta_{-1}) + \lambda_7b(Q_1 - Q_{-1}) \end{pmatrix}$$

where we use notations  $\mathcal{T}^i(\theta) = \theta_i$ ,  $\mathcal{T}^i(\eta) = \eta_i$  for  $i \in \mathbb{Z}$ ,

$$(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}a\theta = P \quad \text{and} \quad (\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b\eta = Q,$$

that is,

$$a\theta = P_1 + P + P_{-1} \quad \text{and} \quad b\eta = Q_1 + Q + Q_{-1}.$$

(The reader should not be confused by the notation in this section with our previous one.  $P_1$  is not the  $\mathfrak{g}_1$  component of  $P$  but rather  $P_1 = \mathcal{T}P$ . Likewise with the other expressions.) We now define the following tri-vector

$$\Psi = \frac{1}{2} \int \xi \wedge \text{Pr}_{\mathcal{H}(\xi)} \mathcal{H} \wedge \xi,$$

where by  $\int$  we mean the equivalent class  $f \equiv g$  iff  $f - g \in \text{Im}(\mathcal{T} - 1)$ . We know that an anti-symmetric operator  $\mathcal{H}$  is Hamiltonian if and only if the tri-vector  $\Theta$  vanishes ([26]). We carry out this computation for  $\mathcal{H}$  in the statement.

$$\begin{aligned} \Psi = \int & (-\lambda_1\theta \wedge b' \wedge \theta_1 + \lambda_2\theta \wedge a' \wedge (P_1 - P_{-1}) + \lambda_5\theta \wedge a' \wedge (Q - Q_{-1}) \\ & - \lambda_5\eta \wedge b' \wedge (P - P_1) - \lambda_6\eta \wedge a' \wedge \eta_{-1} + \lambda_7\eta \wedge b' \wedge (Q_1 - Q_{-1})). \end{aligned}$$

Here  $a'$  takes the value of the first entry of  $\mathcal{H}(\xi)$  and  $b'$  takes the value of its second entry. We substitute them into the above expression. Instead of computing the whole expression of  $\Theta$ , we compute its independent terms, which should all vanish. First let us look at 3-forms involving only  $\theta$ . These terms are

$$\begin{aligned} & \int (-\lambda_1\theta \wedge (\lambda_3\theta_2 - \lambda_4\theta_{-1} - \lambda_5b(P - P_1)) \wedge \theta_1 + \lambda_2\theta \wedge \lambda_1(b_{-1}\theta_{-1} - b\theta_1) \wedge (P_1 - P_{-1})) \\ = & \int (\lambda_1\lambda_4\theta \wedge \theta_{-1} \wedge \theta_1 - \lambda_1\lambda_3\theta \wedge \theta_2 \wedge \theta_1 + \lambda_1\lambda_5b\theta \wedge (P - P_1) \wedge \theta_1 \\ & + \lambda_2\lambda_1b_{-1}\theta \wedge \theta_{-1} \wedge (P_1 - P_{-1}) - \lambda_2\lambda_1b\theta \wedge \theta_1 \wedge (P_1 - P_{-1})) \\ = & \int (-\lambda_1(\lambda_4 - \lambda_3)\theta \wedge \theta_1 \wedge \theta_{-1} + \lambda_1b\theta \wedge \theta_1 \wedge (\lambda_2(P - P_1 + P_{-1} - P_2) - \lambda_5(P - P_1))) \\ = & \lambda_1 \int ((\lambda_4 - \lambda_3)\theta \wedge \theta_1 \wedge \theta_{-1} + (\lambda_2 - \lambda_5)b\theta \wedge \theta_1 \wedge (P - P_1)). \end{aligned}$$

We have used  $a\theta = P_1 + P + P_{-1}$  and  $a_1\theta_1 = P + P_1 + P_2$ . This part vanishes when

$$(45) \quad \lambda_1(\lambda_4 - \lambda_3) = 0, \quad \lambda_1(\lambda_2 - \lambda_5) = 0.$$

Next let us look at 3-forms involving only  $\eta$ . These terms are

$$\begin{aligned}
& \int (-\lambda_6 \eta \wedge (\lambda_4 \eta_1 - \lambda_3 \eta_{-2} + \lambda_5 a(Q - Q_{-1})) \wedge \eta_{-1} \\
& \quad + \lambda_7 \eta \wedge (\lambda_6 (a_1 \eta_1 - a \eta_{-1})) \wedge (Q_1 - Q_{-1})) \\
= & \lambda_6 \int ((\lambda_3 - \lambda_4) \eta \wedge \eta_1 \wedge \eta_{-1} - \lambda_5 a \eta \wedge \eta_{-1} \wedge (Q_{-1} - Q) \\
& \quad - \lambda_7 a \eta \wedge \eta_{-1} \wedge (Q_1 - Q_{-1} + Q - Q_{-2})) \\
= & \lambda_6 \int ((\lambda_3 - \lambda_4) \eta \wedge \eta_{-1} \wedge \eta_1 + (\lambda_7 - \lambda_5) a \eta \wedge \eta_{-1} \wedge (Q_{-1} - Q)).
\end{aligned}$$

We used  $b\eta = Q_1 + Q + Q_{-1}$  and  $b_{-1}\eta_{-1} = Q + Q_{-1} + Q_{-2}$ . These terms vanish whenever

$$(46) \quad \lambda_6(\lambda_4 - \lambda_3) = 0, \quad \lambda_6(\lambda_7 - \lambda_5) = 0.$$

We then look at the terms involving a 2-form in  $\theta$  and a one-form in  $\eta$ .

$$\begin{aligned}
& \int (-\lambda_1 \theta \wedge (\lambda_6 (a_1 \eta_1 - a \eta_{-1}) + \lambda_7 b(Q_1 - Q_{-1})) \wedge \theta_1 \\
& \quad + \lambda_2 \theta \wedge (\lambda_4 \eta_1 - \lambda_3 \eta_{-2} + \lambda_5 a(Q - Q_{-1})) \wedge (P_1 - P_{-1}) \\
& \quad + \lambda_5 \theta \wedge (\lambda_1 (b_{-1} \theta_{-1} - b \theta_1) + \lambda_2 a(P_1 - P_{-1})) \wedge (Q - Q_{-1}) \\
& \quad - \lambda_5 \eta \wedge (\lambda_3 \theta_2 - \lambda_4 \theta_{-1} - \lambda_5 b(P - P_1)) \wedge (P - P_1)) \\
= & \int (a \lambda_1 \lambda_6 \theta \wedge \eta \wedge \theta_{-1} - a \lambda_1 \lambda_6 \theta_1 \wedge \eta_{-1} \wedge \theta - b(\lambda_7 - \lambda_5) \lambda_1 \theta \wedge (Q_1 - Q_{-1}) \wedge \theta_1 \\
& \quad + \lambda_2 \lambda_3 \theta_1 \wedge \eta_{-1} \wedge (P - P_2) - \lambda_4 \lambda_2 \theta_{-1} \wedge \eta \wedge (P_{-2} - P) \\
& \quad + \lambda_5 \lambda_4 \eta \wedge \theta_{-1} \wedge (P - P_1) - \lambda_3 \lambda_5 \eta_{-1} \wedge \theta_1 \wedge (P_{-1} - P)) \\
= & \int (a \lambda_1 \lambda_6 \theta \wedge \eta \wedge \theta_{-1} - a \lambda_1 \lambda_6 \theta_1 \wedge \eta_{-1} \wedge \theta - b(\lambda_7 - \lambda_5) \lambda_1 \theta \wedge (Q_1 - Q_{-1}) \wedge \theta_1 \\
& \quad + \lambda_2 \lambda_3 \theta_1 \wedge \eta_{-1} \wedge (2P + P_1 + P_{-1} - P_{-1}) - \lambda_4 \lambda_2 \theta_{-1} \wedge \eta \wedge (P_1 - P_1 - P_{-1} - 2P) \\
& \quad + \lambda_5 \lambda_4 \eta \wedge \theta_{-1} \wedge (P - P_1) - \lambda_3 \lambda_5 \eta_{-1} \wedge \theta_1 \wedge (P_{-1} - P)) \\
= & \int (a(\lambda_1 \lambda_6 - \lambda_4 \lambda_2) \theta \wedge \eta \wedge \theta_{-1} - a(\lambda_1 \lambda_6 - \lambda_2 \lambda_3) \theta_1 \wedge \eta_{-1} \wedge \theta \\
& \quad - b(\lambda_7 - \lambda_5) \lambda_1 \theta \wedge (Q_1 - Q_{-1}) \wedge \theta_1 \\
& \quad + (\lambda_2 - \lambda_5) \lambda_3 \theta_1 \wedge \eta_{-1} \wedge (P - P_{-1}) - \lambda_4 (\lambda_2 - \lambda_5) \theta_{-1} \wedge \eta \wedge (P_1 - P)).
\end{aligned}$$

These terms vanish if

$$(47) \quad \begin{aligned} & \lambda_1 \lambda_6 - \lambda_2 \lambda_3 = 0, \quad \lambda_1 \lambda_6 - \lambda_4 \lambda_2 = 0, \quad (\lambda_7 - \lambda_5) \lambda_1 = 0, \\ & (\lambda_2 - \lambda_5) \lambda_3 = 0, \quad \lambda_4 (\lambda_2 - \lambda_5) = 0. \end{aligned}$$

Finally, we look at the terms involving a 2-form in  $\eta$  and a one-form in  $\theta$ .

$$\begin{aligned}
& \int (\lambda_5 \theta \wedge (\lambda_4 \eta_1 - \lambda_3 \eta_{-2} + \lambda_5 a(Q - Q_{-1})) \wedge (Q - Q_{-1}) \\
& - \lambda_5 \eta \wedge (\lambda_6(a_1 \eta_1 - a \eta_{-1}) + \lambda_7 b(Q_1 - Q_{-1})) \wedge (P - P_1) \\
& - \lambda_6 \eta \wedge (\lambda_1(b_{-1} \theta_{-1} - b \theta_1) + \lambda_2 a(P_1 - P_{-1})) \wedge \eta_{-1} \\
& + \lambda_7 \eta \wedge (\lambda_3 \theta_2 - \lambda_4 \theta_{-1} - \lambda_5 b(P - P_1)) \wedge (Q_1 - Q_{-1})) \\
= & \int (\lambda_1 \lambda_6 b \eta \wedge \theta_1 \wedge \eta_{-1} - b \lambda_1 \lambda_6 \eta_1 \wedge \theta \wedge \eta - a(\lambda_2 - \lambda_5) \lambda_6 \eta \wedge (P_1 - P_{-1}) \wedge \eta_{-1} \\
& + \lambda_5 \lambda_4 \theta \wedge \eta_1 \wedge (Q - Q_{-1}) - \lambda_5 \lambda_3 \theta_1 \wedge \eta_{-1} \wedge (Q_1 - Q) \\
& + \lambda_7 \lambda_3 \eta_{-1} \wedge \theta_1 \wedge (Q - Q_{-2}) - \lambda_4 \lambda_7 \eta_1 \wedge \theta \wedge (Q_2 - Q)) \\
= & \int (b(\lambda_1 \lambda_6 - \lambda_3 \lambda_7) \eta \wedge \theta_1 \wedge \eta_{-1} - b(\lambda_1 \lambda_6 - \lambda_4 \lambda_7) \eta_1 \wedge \theta \wedge \eta \\
& - a(\lambda_2 - \lambda_5) \lambda_6 \eta \wedge (P_1 - P_{-1}) \wedge \eta_{-1} \\
& + (\lambda_5 - \lambda_7) \lambda_4 \theta \wedge \eta_1 \wedge (Q - Q_{-1}) - (\lambda_5 - \lambda_7) \lambda_3 \theta_1 \wedge \eta_{-1} \wedge (Q_1 - Q)).
\end{aligned}$$

It vanishes when

$$(48) \quad \begin{aligned} \lambda_1 \lambda_6 - \lambda_7 \lambda_3 = 0, & \quad \lambda_1 \lambda_6 - \lambda_4 \lambda_7 = 0, & \quad (\lambda_2 - \lambda_5) \lambda_6 = 0, \\ (\lambda_5 - \lambda_7) \lambda_3 = 0, & \quad (\lambda_7 - \lambda_5) \lambda_4 = 0. \end{aligned}$$

From here we conclude that the operator  $\mathcal{H}$  is Hamiltonian if and only if the parameters satisfy all the conditions written out in (45), (46), (47) and (48). Using the Maple package Gröbner to solve this algebraic system, we can sum up the solutions and obtain the three cases listed in the statement of the theorem.  $\square$

Clearly case (1) in theorem 6.2 is not interesting in our search for integrable systems since the Hamiltonian pair is independent of dependent variables. The same happens to the Hamiltonian operator in case (3) since we can rewrite it as

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \lambda_2(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1} & \lambda_5(1 - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1} \\ -\lambda_5(1 - \mathcal{T})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1} & \lambda_7(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

We now look at case (2). Without losing generality, we can write down two Hamiltonian pairs as stated in the following theorem. For convenience, we write  $a$  and  $b$  as the dependent variables in one pair and  $\tilde{a}$  and  $\tilde{b}$  in another pair.

**Theorem 6.3.** *Let  $\lambda$  be an arbitrary constant. Then, the operators*

$$\mathcal{P}_1 = \begin{pmatrix} 0 & \mathcal{T} - \mathcal{T}^{-2} \\ \mathcal{T}^2 - \mathcal{T}^{-1} & \lambda(\mathcal{T}a - a\mathcal{T}^{-1}) \end{pmatrix}$$

and

$$\mathcal{P}_2 = \begin{pmatrix} \mathcal{T}^{-1}b - b\mathcal{T} & \lambda a(1 - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b \\ +\lambda a(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}a & \lambda b(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b \\ -\lambda b(1 - \mathcal{T})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}a & \lambda b(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})^{-1}b \end{pmatrix}$$

form a Hamiltonian pair, and the operators

$$\mathcal{Q}_1(\tilde{a}, \tilde{b}) = \begin{pmatrix} \lambda(\mathcal{T}^{-1}\tilde{b} - \tilde{b}\mathcal{T}) & \mathcal{T} - \mathcal{T}^{-2} \\ \mathcal{T}^2 - \mathcal{T}^{-1} & 0 \end{pmatrix}$$

and

$$\mathcal{Q}_2(\tilde{a}, \tilde{b}) = \begin{pmatrix} \lambda\tilde{a}(\mathcal{T}-\mathcal{T}^{-1})(\mathcal{T}+1+\mathcal{T}^{-1})^{-1}\tilde{a} & \lambda\tilde{a}(1-\mathcal{T}^{-1})(\mathcal{T}+1+\mathcal{T}^{-1})^{-1}\tilde{b} \\ -\lambda\tilde{b}(1-\mathcal{T})(\mathcal{T}+1+\mathcal{T}^{-1})^{-1}\tilde{a} & \mathcal{T}\tilde{a}-\tilde{a}\mathcal{T}^{-1} \\ & +\lambda\tilde{b}(\mathcal{T}-\mathcal{T}^{-1})(\mathcal{T}+1+\mathcal{T}^{-1})^{-1}\tilde{b} \end{pmatrix}$$

form another Hamiltonian pair. Moreover, the Hamiltonian pair  $\mathcal{P}_1$  and  $\mathcal{P}_2$  is related to the Hamiltonian pair  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  by the Miura transformation  $\tilde{a} = -b$  and  $\tilde{b} = -a_1$ , that is,

$$\mathcal{Q}_1(\tilde{a}, \tilde{b}) = D_{(\tilde{a}, \tilde{b})}\mathcal{P}_1 D_{(\tilde{a}, \tilde{b})}^* \quad \text{and} \quad \mathcal{Q}_2(\tilde{a}, \tilde{b}) = D_{(\tilde{a}, \tilde{b})}\mathcal{P}_2 D_{(\tilde{a}, \tilde{b})}^*.$$

This Miura transformation is induced by projective duality.

*Proof.* The first part of the proof of this statement is straightforward. They are two Hamiltonian pairs as a direct result of case (2) in theorem 6.2 and the relation between them can be checked by a simple calculation. To see that the Miura transformation is induced by projective duality, we will use the map  $\alpha$  defined in [27]. (Recall that  $(a, b)$  in [27] is in fact  $(\hat{a}, \hat{b})$  here, a difference created by the gauge transformation we introduced initially and the choice of a right frame, versus de left one used in [27].) Define  $\alpha : \mathbb{RP}^2 \rightarrow (\mathbb{RP}^2)^*$  as

$$\alpha(x_i) = \overline{x_i x_{i+1}},$$

where  $\overline{x_i x_{i+1}}$  is the line joining  $x_i$  and  $x_{i+1}$ . The authors of [27] showed that  $\alpha^*(\hat{a}_i) = -\hat{b}_{i+1}$  and  $\alpha^*(\hat{b}_i) = -\hat{a}_i$  where if  $(\hat{a}_i, \hat{b}_i)$  are the invariants for  $x_i$ ,  $(\alpha^*(\hat{a}_i), \alpha^*(\hat{b}_i))$  are the projective invariants associated to the polygon whose vertices are the points dual to  $\alpha(x_i)$ . Given that

$$\hat{a}_i = -a_{i+1}, \quad \hat{b}_i = -b_i$$

we have

$$\alpha^*(a_i) = -b_i, \quad \alpha^*(b_i) = -a_{i+1}.$$

□

From here it follows that, to study integrable systems associated to biHamiltonian structures included in  $\mathcal{P}$ , we only need to derive biHamiltonian integrable systems for the Hamiltonian pair  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Based on a difference analogue to the Adler Residue Theorem ([1, 25]), we conclude that the first Hamiltonian associated to this pair is  $\ln b$ . This leads to the following bi-Hamiltonian system

$$(49) \quad \begin{cases} a_t = \frac{1}{b_1} - \frac{1}{b_{-2}} \\ b_t = \lambda \left( \frac{a_1}{b_1} - \frac{a}{b_{-1}} \right) \end{cases}$$

with

$$(50) \quad \begin{pmatrix} a \\ b \end{pmatrix}_t = \mathcal{P}_1 \delta f = \mathcal{P}_2 \delta g, \quad \text{where} \quad f = \ln b \quad \text{and} \quad g = -\frac{a_1}{bb_1}.$$

Notice that the constant  $\lambda$  in (49) can be scaled away by a simple scaling transformation  $a \mapsto \frac{a}{\lambda^{1/3}}$  and  $b \mapsto \lambda^{1/3}b$  if  $\lambda \neq 0$ . We will keep it instead of taking  $\lambda = 1$ .

Let us introduce the following transformation ([13])

$$(51) \quad u = \frac{1}{bb_1b_2}, \quad v = -\frac{a_1}{bb_1}.$$

Their Fréchet derivatives with respect to  $a$  and  $b$  are given by

$$D_{(u,v)} = \begin{pmatrix} 0 & -u(1 + \mathcal{T} + \mathcal{T}^2)\frac{1}{b} \\ v\mathcal{T}\frac{1}{a} & -v(1 + \mathcal{T})\frac{1}{b} \end{pmatrix} = \begin{pmatrix} 0 & u(1 + \mathcal{T} + \mathcal{T}^2) \\ v\mathcal{T} & v(1 + \mathcal{T}) \end{pmatrix} \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & -\frac{1}{b} \end{pmatrix}.$$

Therefore, when we change the variable the Hamiltonian operator  $\mathcal{P}_1$  transforms into

$$\begin{aligned} \tilde{\mathcal{P}}_1 &= D_{(u,v)}\mathcal{P}_1D_{(u,v)}^* \\ &= \begin{pmatrix} 0 & u(1 + \mathcal{T} + \mathcal{T}^2) \\ v\mathcal{T} & v(1 + \mathcal{T}) \end{pmatrix} \begin{pmatrix} 0 & \frac{u-1}{v-1}\mathcal{T} - \mathcal{T}^{-2}\frac{u}{v-1} \\ \frac{u}{v-1}\mathcal{T}^2 - \mathcal{T}^{-1}\frac{u-1}{v-1} & \lambda(\mathcal{T}^{-1}v - v\mathcal{T}) \end{pmatrix} \\ &\quad \begin{pmatrix} 0 & \mathcal{T}^{-1}v \\ (1 + \mathcal{T}^{-1} + \mathcal{T}^{-2})u & (1 + \mathcal{T}^{-1})v \end{pmatrix} \\ &= \begin{pmatrix} 0 & u(1 + \mathcal{T} + \mathcal{T}^2)(u\mathcal{T} - \mathcal{T}^{-2}u) \\ (u\mathcal{T}^2 - \mathcal{T}^{-1}u)(1 + \mathcal{T}^{-1} + \mathcal{T}^{-2})u & v(1 + \mathcal{T})(u\mathcal{T} - \mathcal{T}^{-2}u) \\ & + (u\mathcal{T}^2 - \mathcal{T}^{-1}u)(1 + \mathcal{T}^{-1})v \end{pmatrix} \\ &+ \lambda \begin{pmatrix} u(1 + \mathcal{T} + \mathcal{T}^2) & 0 \\ 0 & v(1 + \mathcal{T}) \end{pmatrix} (\mathcal{T}^{-1}v - v\mathcal{T}) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (1 + \mathcal{T}^{-1} + \mathcal{T}^{-2})u & 0 \\ 0 & (1 + \mathcal{T}^{-1})v \end{pmatrix}. \end{aligned}$$

Similarly, the Hamiltonian operator  $\mathcal{P}_2$  changes into

$$\tilde{\mathcal{P}}_2 = \begin{pmatrix} \lambda u(\mathcal{T} - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})u & \lambda u(\mathcal{T} - 1)(\mathcal{T} + 1 + \mathcal{T}^{-1})v \\ \lambda v(1 - \mathcal{T}^{-1})(\mathcal{T} + 1 + \mathcal{T}^{-1})u & \lambda v(\mathcal{T} - \mathcal{T}^{-1})v + \mathcal{T}^{-1}u - u\mathcal{T} \end{pmatrix}.$$

Notice that *under this transformation both operators are local*. This Hamiltonian pair can be found in [3]. The biHamiltonian system (49) becomes

$$(52) \quad \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} \lambda u(v_2 - v_{-1}) \\ u_{-1} - u + \lambda v(v_1 - v_{-1}) \end{pmatrix} = \tilde{\mathcal{P}}_1\delta f = \tilde{\mathcal{P}}_2\delta g,$$

where  $f = -\frac{1}{3}\ln u$  and  $g = v$ . When  $\lambda = 1$  (as mentioned before, we can scale  $\lambda$  to 1 when  $\lambda \neq 0$ ), this system has appeared in [13], where the authors studied the integrable systems related the lattice  $W$ -algebras. It is the Boussinesq lattice related to the lattice  $W_3$ -algebra.

Notice that  $\mathcal{P}_1|_{\lambda=1} + \mathcal{P}_2|_{\lambda=1} = \mathcal{P}$ , where the operator  $\mathcal{P}$  is defined by (35). Notice also that  $\mathcal{P}_2(\ln b) = 0$ . Therefore, for the evolution induced on the invariants by invariant evolutions of planar polygons, we have the following result:

**Theorem 6.4.** *Let  $f(a, b) = \ln b$  and let  $\pi : \mathbb{R}^3 \rightarrow \mathbb{RP}^2$  be the projection associated to the lift  $x \rightarrow \begin{pmatrix} x \\ 1 \end{pmatrix}$ . Then, the evolution induced on the invariants  $a$  and  $b$  by the invariant evolution of planar polygons*

$$(x_s)_t = \pi \left( \frac{1}{b_s} V_{s+2} + \frac{a_s}{b_s} V_{s+1} + V_s \right)$$

*is the biHamiltonian equation*

$$(53) \quad \begin{pmatrix} a \\ b \end{pmatrix}_t = \begin{pmatrix} \frac{1}{b_1} - \frac{1}{b_{-2}} \\ \frac{a_1}{b_1} - \frac{a}{b_{-1}} \end{pmatrix} = \mathcal{P}\delta \ln b = \mathcal{P}_2|_{\lambda=1}\delta \left( -\frac{a_1}{bb_1} \right).$$

*Under the Miura transformation (51), it is transformed into the Boussinesq lattice related to the lattice  $W_3$ -algebra*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u(v_2 - v_{-1}) \\ u_{-1} - u + v(v_1 - v_{-1}) \end{pmatrix}.$$

Based on this theorem, we can anticipate that integrable discretizations of  $W_n$ -algebras should be induced by invariant evolutions of projective polygons in  $\mathbb{RP}^n$ . In our last section we describe the  $\mathbb{RP}^n$  generalization and prove it is integrable. But first we give a further reduction producing yet another integrable system.

Indeed, the next symmetry flow of equation (53) is

$$\begin{pmatrix} a \\ b \end{pmatrix}_\tau = \mathcal{P}_1|_{\lambda=1} \delta \left( -\frac{a_1}{bb_1} \right) = \left( \frac{\frac{a_2}{b_1^2 b_2} + \frac{a_1}{bb_1^2} - \frac{a_{-1}}{b_{-2}^2 b_{-1}} - \frac{a_{-2}}{b_{-2}^2 b_{-3}}}{\frac{1}{b_{-1} b_{-2}} - \frac{1}{b_1 b_2} + \frac{a_1}{b_1^2} \left( \frac{a_2}{b^2} + \frac{a_1}{b} \right) - \frac{a}{b_{-1}^2} \left( \frac{a}{b} + \frac{a_{-1}}{b_{-2}} \right)} \right).$$

Here we use the time variable  $\tau$  instead of  $t$  to avoid confusion. It admits a reduction

$$b_\tau = \frac{1}{b_{-1} b_{-2}} - \frac{1}{b_1 b_2},$$

when  $a = 0$ . Under the Miura transformation (51), it becomes

$$u_\tau = u(u_1 + u_2 - u_{-1} - u_{-2}).$$

This is the Narita-Itoh-Bogoyavlensky lattice [4]

$$u_t = u \left( \sum_{i=1}^p u_i - \sum_{i=1}^p u_{-i} \right)$$

with  $p = 2$ . In general, the Narita-Itoh-Bogoyavlensky lattice is equivalent to the lattice  $W_{p+1}$  algebra [13]. Recently, the biHamiltonian structures for the Narita-Itoh-Bogoyavlensky lattice were constructed using the Lax representation [35]. Notice that there is no reduction for the corresponding invariant evolution of planar polygons since

$$\mathcal{P} \delta \left( -\frac{a_1}{bb_1} \right) = (\mathcal{P}_1|_{\lambda=1} + \mathcal{P}_2|_{\lambda=1}) \delta \left( -\frac{a_1}{bb_1} \right) = \begin{pmatrix} a \\ b \end{pmatrix}_\tau + \begin{pmatrix} a \\ b \end{pmatrix}_t.$$

To finish this part, notice that the Poisson structure used in [27] is not compatible with the ones found here, a fact that was also pointed out in [16]. Still, here we have a variety of brackets and one ideally would check the behavior of the pentagram map with respect to any of them to understand better their relation.

**6.2. Hamiltonian pencils and completely integrable systems in  $\mathbb{RP}^n$ .** In our last section we wonder about the origins of the pencil in Theorem 6.3 and we try to generalize it, together with the planar integrable system. In particular, let us define the right bracket on  $G^N$  as in (12)

$$\{\mathcal{F}, \mathcal{G}\}'(L) = \hat{r}(\nabla' \mathcal{F}(L) \wedge \nabla' \mathcal{G}(L)).$$

Even if we were to use an  $R$ -matrix  $r$  for its definition instead of  $\hat{r}$ , this (or the left) bracket is in general not Poisson, and one can easily check that Jacobi's identity is not satisfied if  $G = \text{SL}(2)$ , for example. Still, we will next show that when it is evaluated on extensions along  $\mathcal{K}$ , the resulting bracket is Poisson. In the particular case of  $\mathbb{RP}^2$ , we in fact obtain  $\frac{1}{2} \mathcal{P}_1|_{\lambda=1}$ , which forms a Hamiltonian pencil with our original reduction. We work out this example first.

**Example 6.5.**

Let  $f, g : \mathcal{K} \rightarrow \mathbb{R}$  be two functions and let  $\mathcal{F}, \mathcal{G}$  be extensions satisfying (37). Then

$$\{f, g\}_0(a, b) = \{\mathcal{F}, \mathcal{G}\}'(K) = \frac{1}{2} \left\langle \begin{pmatrix} \mathcal{T}^{-1} f_b \\ \mathcal{T}^{-1} f_a \end{pmatrix}, \begin{pmatrix} g_{12} + a g_b \\ g_b \end{pmatrix} \right\rangle - \frac{1}{2} \begin{pmatrix} \mathcal{T}^{-1} g_b \\ \mathcal{T}^{-1} g_a \end{pmatrix}, \begin{pmatrix} f_{12} + a f_b \\ f_b \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} f_a & f_b \end{pmatrix} \begin{pmatrix} 0 & \mathcal{T} - \mathcal{T}^{-2} \\ \mathcal{T}^2 - \mathcal{T}^{-1} & \mathcal{T}a - a\mathcal{T}^{-1} \end{pmatrix} \begin{pmatrix} g_a \\ g_b \end{pmatrix} = \frac{1}{2} \nabla f^T \mathcal{P}_1|_{\lambda=1} \nabla g.$$

Indeed, this evaluation produces a Poisson bracket for any dimension.

**Proposition 6.6.** *The evaluation of (12) along  $\mathcal{K}$  obtained by the formula*

$$\{f, g\}_0(\mathbf{k}) = \{\mathcal{F}, \mathcal{G}\}'(K)$$

is defined explicitly by a  $n \times n$  operator matrix  $\mathcal{H}$  whose  $(i, j)$  entry is equal to:

- (1) zero if  $i + j > n + 1$ ;
- (2)  $\mathcal{T}^j - \mathcal{T}^{-i}$  if  $i + j = n + 1$
- (3)  $\mathcal{T}^j k^{i+j} - k^{i+j} \mathcal{T}^{-i}$  if  $i + j < n + 1$ .

*Proof.* This bracket can, in fact, be found explicitly. Assume

$$(54) \quad \nabla_s \mathcal{F}(K) = \begin{pmatrix} Q_s & \frac{\partial f}{\partial \mathbf{k}_s} \\ \mathbf{q}_s^T & -\text{tr}(Q_s) \end{pmatrix}$$

and assume  $Q_s = (q_{ij}^s)$  and  $\mathbf{q}_s = (q_i^s)$ . Let us call  $\mathbf{f} = \frac{\partial f}{\partial \mathbf{k}_s} := (f^i)$  and  $\varpi = -\text{tr}(Q_s)$ . As before, we drop the subindex  $s$  to avoid cluttering. With this notation, and since

$$K = \begin{pmatrix} I_n & 0 \\ \mathbf{k}^T & 1 \end{pmatrix} \Lambda$$

where

$$\Lambda = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

we can write

$$(55) \quad \nabla' \mathcal{F} = \Lambda^{-1} \begin{pmatrix} Q + \mathbf{f} \mathbf{k}^T & \mathbf{f} \\ \mathbf{q}^T - \mathbf{k}^T Q + (\varpi - \mathbf{k}^T \mathbf{f}) \mathbf{k}^T & \varpi - \mathbf{k}^T \mathbf{f} \end{pmatrix} \Lambda.$$

Next we notice that conjugation by  $\Lambda$  shifts rows once up and columns once to the left (with the first ones moving to last position, after multiplication by  $(-1)^n$ ). That means the evaluation of (12) is explicitly given by

$$(56) \quad \{f, g\}_0(\mathbf{k}) = \frac{(-1)^n}{2} \langle \mathcal{T}^{-1} \mathbf{f}, \begin{pmatrix} q_{12}^g \\ \vdots \\ q_{1n}^g \\ g^1 \end{pmatrix} + g^1 \begin{pmatrix} k^2 \\ \vdots \\ k^n \\ 0 \end{pmatrix} \rangle - \frac{(-1)^n}{2} \langle \mathcal{T}^{-1} \mathbf{g}, \begin{pmatrix} q_{12}^f \\ \vdots \\ q_{1n}^f \\ f^1 \end{pmatrix} + f^1 \begin{pmatrix} k^2 \\ \vdots \\ k^n \\ 0 \end{pmatrix} \rangle$$

We will know the bracket explicitly once we find  $q_{1j}$ ,  $j = 2, \dots, n$ , as we do next. Equation (37) implies that the first  $n$  rows of (37) are zero. In our notation

$$\begin{aligned} \mathcal{T}^{-1} q_{ij}^f &= q_{i+1, j+1}^f + f^{i+1} k^{j+1}, \quad j = 1, \dots, n-1, \\ \mathcal{T}^{-1} q_{in}^f &= f^{i+1}, \quad i = 1, \dots, n-1, \\ \mathcal{T}^{-1} f_i &= (-1)^n (q_{i+1, 1} + f^{i+1} k^i). \end{aligned}$$

From here we have  $q_{1n} = \mathcal{T} f^2$  and

$$(57) \quad q_{1j} = \mathcal{T}^{n-j+1} f^{n-j+2} + \sum_{p=2}^{n-j+1} \mathcal{T}^{p-1} (f^p k^{j-1+p})$$

for any  $j = 2, 3, \dots, n-1$ . Substituting these values in (56) we get the expression in the statement of the proposition.  $\square$

**Theorem 6.7.** *The structure defined in proposition 6.6 is a Poisson structure.*

*Proof.* We will prove that if  $\mathcal{H}$  is the operator defined by the bracket in proposition 6.6, then  $\mathcal{H}$  is Hamiltonian. To this end we will use a theorem analogous to theorem 7.8 in [26] for difference operators and following a process similar to the proof of Theorem 6.2. Let  $\theta$  be a column vector with entry  $\theta^i$ . For the operator  $\mathcal{H}$ , we have  $\mathcal{H}(\theta)$  to be a column vector with entry

$$\mathcal{H}(\theta)^i = \theta_{n+1-i}^{n+1-i} - \theta_{-i}^{n+1-i} + \sum_{j=1}^{n-i} \left( (k^{i+j}\theta^j)_j - k^{i+j}\theta_{-i}^j \right).$$

We now define the bi-vectors

$$\begin{aligned} \Theta &= \frac{1}{2} \int \theta \wedge \mathcal{H}(\theta) \\ &= \frac{1}{2} \sum_{i=1}^n \int \left( \theta^i \wedge (\theta_{n+1-i}^{n+1-i} - \theta_{-i}^{n+1-i}) + \theta^i \wedge \sum_{j=1}^{n-i} \left( (k^{i+j}\theta^j)_j - k^{i+j}\theta_{-i}^j \right) \right) \\ &= \sum_{i=1}^n \int \theta^i \wedge \theta_{n+1-i}^{n+1-i} + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \int k^{i+j}\theta_{-j}^i \wedge \theta^j. \end{aligned}$$

We know that an anti-symmetric operator  $\mathcal{H}$  is Hamiltonian if and only if the tri-vector  $\text{Pr}_{\mathcal{H}(\theta)}\Theta$  vanishes (see [26]). We now show this is the case for the given  $\mathcal{H}$ . Indeed, we have

$$\begin{aligned} \text{Pr}_{\mathcal{H}(\theta)}\Theta &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \int \mathcal{H}(\theta)^{i+j}\theta_{-j}^i \wedge \theta^j \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \int \left( \theta_{n+1-i-j}^{n+1-i-j} - \theta_{-i-j}^{n+1-i-j} \right) \wedge \theta_{-j}^i \wedge \theta^j \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{l=1}^{n-i-j} \int \left( (k^{i+j+l}\theta^l)_l - k^{i+j+l}\theta_{-i-j}^l \right) \wedge \theta_{-j}^i \wedge \theta^j \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \int \left( \theta_{n+1-i-j}^{n+1-i-j} \wedge \theta_{-j}^i \wedge \theta^j - \theta_{-i}^{n+1-i-j} \wedge \theta^i \wedge \theta_{-j}^j \right) \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{l=1}^{n-i-j} k^{i+j+l} \int \left( \theta^l \wedge \theta_{-j-l}^i \wedge \theta_{-l}^j - \theta_{-i-j}^l \wedge \theta_{-j}^i \wedge \theta^j \right) \\ &= 0 \end{aligned}$$

by changing dummy variables. Thus we proved the statement.  $\square$

Using the general reduced operator we would like to write down an  $n$ -component integrable system. Notice that we have not proved in general that the two  $n$ -dimensional reduced brackets form a pencil. Nevertheless, we will be able to find an integrable system induced on  $\mathbf{k}$  by an invariant evolution of polygons in  $\mathbb{RP}^n$ . First we notice the following fact.



**Proposition 6.8.** *The two functionals  $\ln k^1$  and  $\frac{k_1^2}{k^1 k_1^1}$  are in involution with respect to the Hamiltonian operator defined in proposition 6.6; that is,*

$$\{\ln k^1, \frac{k_1^2}{k^1 k_1^1}\}_0 = \int (\delta \ln k^1)^T \mathcal{H} \delta \frac{k_1^2}{k^1 k_1^1} = 0.$$

*Proof.* This can be proved by direct calculations. First we have

$$\begin{aligned} \delta \ln k^1 &= \left( \frac{1}{k^1}, 0, \dots, 0 \right)^T; \\ \delta \frac{k_1^2}{k^1 k_1^1} &= \left( -\frac{k_1^2}{(k^1)^2 k_1^1} - \frac{k^2}{(k^1)^2 k_{-1}^1}, \frac{1}{k^1 k_{-1}^1}, 0, \dots, 0 \right)^T, \end{aligned}$$

which leads to

$$\left( \delta \frac{k_1^2}{k^1 k_1^1} \right)^T \mathcal{H} \delta \ln k^1 = -(\mathcal{T} - 1) \left( \frac{k^2}{k^1 k_{-1}^1} \right)^2$$

and thus we obtain the result in the statement.  $\square$

From this proposition, we know that the corresponding Hamiltonian vector fields commute, that is,  $[\mathcal{H} \delta \ln k^1, \mathcal{H} \delta \frac{k_1^2}{k^1 k_1^1}] = 0$ .

Finally, we are going to show that the Hamiltonian system  $\mathbf{k}_t = \mathcal{H} \delta \ln k^1$  is integrable. Here we say a system is integrable if it possesses a hierarchy of infinitely many commuting symmetries. We first write the system explicitly as

$$(58) \quad \begin{cases} k_t^i = \frac{k_1^{i+1}}{k_1^1} - \frac{k^{i+1}}{k_{-1}^1}, & i = 1, 2, \dots, n-1 \\ k_t^n = \frac{1}{k_1^1} - \frac{1}{k_{-n}^1} \end{cases}$$

We then introduce the Miura transformation

$$u^1 = \frac{1}{k^1 k_1^1 \dots k_n^1}, \quad u^i = \frac{k_{i-1}^i}{k^1 k_1^1 \dots k_{i-1}^1}, \quad i = 2, 3, \dots, n.$$

Under this transformation, equation (58) becomes  $\mathbf{u}_t = -\tilde{\mathcal{H}} \delta \frac{\ln u^1}{n+1}$ , where  $\tilde{\mathcal{H}} = D_{\mathbf{u}} \mathcal{H} D_{\mathbf{u}}^*$  denote the transformed Hamiltonian operator. It is written as

$$(59) \quad \begin{cases} u_t^1 = -u^1(u_n^2 - u_{-1}^2) \\ u_t^i = u^{i+1} - u_{-1}^{i+1} - u^i(u_{i-1}^2 - u_{-1}^2), & i = 2, 3, \dots, n-1 \\ u_t^n = u^1 - u_{-1}^1 - u^n(u_{n-1}^2 - u_{-1}^2) \end{cases}$$

If we introduce the convention that  $u^j = u^{j-n}$  if  $j > n$ , we can put the last two equations together as

$$u_t^i = u^{i+1} - u_{-1}^{i+1} - u^i(u_{i-1}^2 - u_{-1}^2), \quad i = 2, 3, \dots, n, \quad (u^{n+1} = u^1).$$

Take  $n = 2$ ,  $u^1 = u$  and  $u^2 = -v$ . Then equation (59) becomes the Boussinesq lattice related to the lattice  $W_3$ -algebra (see (52)).

We observe that the vector  $\tau = (\tau^1, \dots, \tau^n)^T$  defined by

$$\begin{aligned} \tau^1 &= s u_t^1 - u^1 \left( (n+1)u_n^2 + \sum_{l=0}^n u_l^2 \right); \\ \tau^i &= s u_t^i - u^i \left( i u_{i-1}^2 + \sum_{l=0}^{i-1} u_l^2 \right) + (i+1)u^{i+1}, \quad i = 2, 3, \dots, n, \end{aligned}$$

where  $s$  is the independent discrete variable and we used the above convention  $u^{n+1} = u^1$ , is a master symmetry (see [10] for definition) of the system (59). Indeed,

we have  $[\mathbf{u}_t, \tau] \neq 0$  and  $[\mathbf{u}_t, [\mathbf{u}_t, \tau]] = 0$ . Thus, we can recursively generate the hierarchy of commuting symmetry flows of system (59) by setting

$$(60) \quad Q^0 = \mathbf{u}_t \quad \text{and} \quad Q^i = [\tau, Q^{i-1}].$$

Therefore, the system (59) is integrable.

**Proposition 6.9.** *The master symmetry is a Hamiltonian vector with Hamiltonian function  $-\frac{2s+n+2}{2(n+1)} \ln u^1$ . Moreover, all symmetries generated by (60) are Hamiltonian vector fields. Their Hamiltonians are given by*

$$f^i = \langle \tau, \delta f^{i-1} \rangle \quad \text{and} \quad f^0 = -\frac{\ln u^1}{n+1}.$$

*Proof.* By straightforward calculation, we can check  $\tau = -\frac{1}{n+1} \tilde{\mathcal{H}} \delta \left( \frac{2s+n+2}{2} \ln u^1 \right)$ . Therefore, the Hamiltonian operator  $\tilde{\mathcal{H}}$  is conserved along the vector field  $\tau$  (see [5]). In a word, the Lie derivative of  $\tilde{\mathcal{H}}$  along  $\tau$  vanished, that is,  $L_\tau \tilde{\mathcal{H}} = 0$ . We know  $Q^0 = \tilde{\mathcal{H}} f^0$ . Take the Lie derivative along the vector field  $\tau$  on its both sides. It follows that

$$Q^1 = L_\tau Q^0 = L_\tau \left( \tilde{\mathcal{H}} \delta f^0 \right) = \tilde{\mathcal{H}} L_\tau \delta f^0 = \tilde{\mathcal{H}} \langle \tau, \delta f^0 \rangle$$

We denote its Hamiltonian as  $f^1$ . So  $Q^1$  is a Hamiltonian vector field with Hamiltonian  $f^1$ . By induction, we can prove the statement.  $\square$

Indeed, we can compute the following Hamiltonians

$$\begin{aligned} f^1 &= -\int \frac{\tau^1}{(n+1)u^1} = \frac{1}{n+1} \int \left( s(u_n^2 - u_{-1}^2) + \left( (n+1)u_n^2 + \sum_{l=0}^n u_l^2 \right) \right) \\ &= u^2 \\ f^2 &= \int \tau^2 = \int \left( s u_t^2 - u^2 \left( 2u_1^2 + \sum_{l=0}^1 u_l^2 \right) + 3u^3 \right) \\ &= 2u^3 - 2u^2 u_1^2 - (u^2)^2 \\ &\quad \dots \end{aligned}$$

These are the integrals obtained in [12] for the classical lattice  $W_{n+1}$  algebra. Thus equation (59) is the integrable lattice related to the lattice  $W_{n+1}$ -algebra, which is not explicitly written down in [12] for all  $n$ .

To finish the paper we will describe the projective realization of the  $\mathbb{RP}^n$  completely integrable system. First a lemma that facilitates the description.

**Lemma 6.10.** *Let  $\mathcal{H}$  be the Hamiltonian operator defined in proposition 6.6, and let  $\mathcal{P}$  be the one given by our original reduction. Then*

$$\mathcal{H}(\delta \ln k^1) = -\mathcal{P}(\delta \ln k^1)$$

*Proof.* From the reduction process, if we want to find  $\mathcal{P}\left(\frac{\partial f}{\partial \mathbf{k}}\right) = \mathcal{P}\left((-1)^n \frac{1}{k^1} e_1\right)$ , we need to find

$$(61) \quad \nabla \mathcal{F} - \mathcal{T} \nabla' \mathcal{F} = \begin{pmatrix} 0 & 0 \\ \mathcal{P}(\mathbf{f})^T & 0 \end{pmatrix}$$

where  $\mathbf{f} = \frac{(-1)^n}{k^1} e_1$  and  $\mathcal{F}$  is a proper extension of  $f(\mathbf{k}) = \ln k^1$ . Using (54) and (55) we can transform this equality into

$$\Lambda \nabla \mathcal{F} - \mathcal{T} \begin{pmatrix} 1 & 0 \\ -\mathbf{k}^T & 1 \end{pmatrix} \nabla \mathcal{F} \begin{pmatrix} 1 & 0 \\ \mathbf{k}^T & 1 \end{pmatrix} \Lambda = \begin{pmatrix} (-1)^n \mathcal{P}(\mathbf{f}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Expression (61) can be written in more detail as

$$(62) \quad \begin{pmatrix} (-1)^n \mathbf{q}^T & (-1)^n \varpi \\ Q & \mathbf{f} \end{pmatrix} - \begin{pmatrix} \mathcal{T}\tilde{Q} + \mathcal{T}(f_1 \tilde{\mathbf{k}}^T) & \mathcal{T}\mathbf{f} & (-1)^n (\mathcal{T}Qe_1 + \mathcal{T}(k^1 \mathbf{f})) \\ \mathcal{T}\tilde{q}^T - \mathcal{T}(\tilde{\mathbf{k}}^T \tilde{Q}) + \mathcal{T}((\varpi - k \cdot \mathbf{f}) \tilde{\mathbf{k}}^T) & \mathcal{T}(\varpi - \mathbf{k} \cdot \mathbf{f}) & (-1)^n (\mathcal{T}q_1 - \mathcal{T}\mathbf{k}^T Qe_1 + \mathcal{T}((\varpi - \mathbf{k} \cdot \mathbf{f})k_1)) \end{pmatrix} \\ = \begin{pmatrix} (-1)^n \mathcal{P}(\mathbf{f}) & 0 \\ 0 & 0 \end{pmatrix}$$

where the tilde indicates that the first column has been removed. From here we get

$$(63) \quad \mathcal{P}(\mathbf{f}) = \mathbf{q} - (-1)^n \begin{pmatrix} \mathcal{T}(\overline{Q^T e_1}) + \mathcal{T}(f_1 \bar{\mathbf{k}}) \\ \mathcal{T}f_1 \end{pmatrix},$$

where the bar indicates that the first row has been removed. Recall from (57) that if  $Q = (q_{ij})$  and  $\mathbf{f} = \frac{(-1)^n}{k^1} e_1$ , then  $q_{1j} = 0$  for  $j \neq 1$ . This means  $\overline{Q^T e_1} = 0$ . With straightforward calculations we can also get that  $q_{11} = -\frac{1}{n+1}$ . Using

$$\begin{aligned} q_1 - \mathbf{k}^T Qe_1 + (\varpi - 1)k_1 &= 0 \\ (-1)^n (\mathcal{T}\overline{Qe_1} + \mathcal{T}\overline{k_1 \mathbf{f}}) &= \frac{(-1)^n}{k^1} e_1, \end{aligned}$$

we obtain

$$\varpi - 1 = -\frac{1}{n+1}, \quad Qe_1 = -\frac{1}{n+1} e_1 + \mathcal{T}^{-1} \frac{1}{k^1} e_2, \quad q_1 = k^2 \mathcal{T}^{-1} \frac{1}{k^1}.$$

A recursive use of the lower left block in (62) produces the values of  $Qe_s$ . They are given by

$$Qe_s = -\frac{1}{n+1} e_s + \mathcal{T}^{-1} \frac{1}{k^1} e_{s+1}, \quad r = 1, \dots, n-1, \quad Qe_n = -\frac{1}{n+1} e_n.$$

The last row of that block gives the value of  $\mathbf{q}$ . They are given by

$$q_i = k^{i+1} \mathcal{T}^{-i} \frac{1}{k^1}.$$

Substituting the values in (63) proves the lemma.  $\square$

Assume  $V_s$  are the original lifts of our projective polygon and  $\hat{k}^i$  are the invariants given by the relation  $V_{s+n+1} = \hat{k}_s^n V_{s+n} + \dots + \hat{k}_s^1 V_{s+1} + (-1)^n V_s$ .

**Theorem 6.11.** *The projectivization of the evolution*

$$(V_s)_t = \frac{-1}{\hat{k}_s^1} \left( V_{s+n} - \hat{k}_{s-1}^n V_{s+n-1} - \dots - \hat{k}_{s-1}^2 V_{s+1} \right) + v_s^0 V_s$$

induces the completely integrable system (58) on the gauged invariants  $\mathbf{k}$ .

*Proof.* Since the  $\mathcal{P}$ -Hamiltonian for the system is  $f(\mathbf{k}) = \ln k^1$ , we know that the lift of the projective realization is given by

$$(V_s)_t = \rho_s^{-1} \mathbf{v}_s$$

where  $\mathbf{v}_s = \frac{\partial f}{\partial \mathbf{k}} = \frac{(-1)^n}{k^1} e_1$ . Since  $\rho_s = (W_{s+n}, \dots, W_s)$  as in (22), we have

$$(V_s)_t = \frac{(-1)^n}{k_s^1} W_{s+n} + v_s^0 V_s.$$

Checking the gauge carefully we see that  $k^1 = (-1)^{n-1} \hat{k}^1$  and  $W_{n+r} = V_{n+r} - \hat{k}_{s-1}^n V_{n+r-1} - \dots - \hat{k}_{s-1}^2 V_{s+1}$ , which concludes the proof.  $\square$

It is only natural to conjecture that both of our reductions form a Hamiltonian pencil associated to integrable discretizations of  $W_n$  algebras. The fact that they are not Poisson brackets before reduction seriously complicates the proof of this conjecture. Nevertheless, it would be very interesting to fully understand the role of the Sklyanin and right (or left) brackets with parabolic gradations in the generation of these Hamiltonian structures, especially since  $R$ -matrices associated to parabolic gradations, even though they are classified in [2], are notoriously hard to use in practice. Further work in this direction is under way.

A final comment on the relationship between our Poisson bracket and the discrete Drinfel'd and Sokolov (DS) reduction described in [9] and [32]. In these two papers the authors described a discrete analogue to the original DS reduction published in [7]. The DS reduced Hamiltonians have been proved ([17]) to have a realization as invariant evolutions of projective curves, thus, it would be natural to assume that the bracket in [9] is equal to ours and it is linked to evolutions of projective polygons. But that is not the case: [9] uses an  $R$ -matrix perturbation defined by the Cartan subalgebra components of the gradients, a perturbation that is needed for its reduction. The author of [16] states that the perturbation vanishes upon reduction, but we believe it does not, and it is in fact fundamental for the bracket in [9]. (This point was also affirmed by Prof. Semenov-Tian-Shansky in an e-mail exchange.) Thus, we believe that both brackets are different, and, unlike the continuous case, the bracket in [9] is not naturally related to polygon evolutions. One needs a unperturbed bracket to have the relation we have shown here.

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