PROJECTIVE-TYPE DIFFERENTIAL INVARIANTS FOR CURVES AND THEIR ASSOCIATED PDES OF KDV TYPE

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Abstract. In this paper we present an overview of the direct relation between differential invariants of projective type for curves in flat semisimple homegenous spaces and PDEs of KdV type. We describe the progress in the proof of a conjectured Theorem stating that for any such space there are geometric evolutions of curves that induce completely integrable evolutions on their invariants of projective-type. The Theorem also states that these evolutions decouple always into either complexly coupled KdV equations (conformal type), decoupled KdV equations (Lagrangian type) or Adler-Gelfand-Dikii generalized KdV types (projective type). The paper also describes the fundamental role of group-based moving frames in this study.

Key words. Completely integrable systems, moving frames, projective differential invariants, curves in semisimple homogeneous spaces, differential invariants of curves, infinite dimensional Poisson geometry.

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1. Moving frames and differential invariants of projective-type for curves in G/H. The classical concept of moving frame was developed by Élie Cartan ([2], [3]). A classical moving frame along a curve in a manifold M is a curve in the frame bundle of the manifold over the curve, invariant under the action of the transformation group under consideration. This method is a very powerful tool but its explicit application relied on intuitive choices that were not clear on a general setting. Some ideas in Cartan's work and later work of Griffiths ([7]), Green ([8]) and others laid the foundation for the concept of a group-based moving frame, that is, an equivariant map between the jet space of curves in the manifold and the group of transformations. Recent work by Fels and Olver ([4, 5]) finally gave the precise definition of the group based moving frame and extended its application beyond its original geometric picture to an astonishingly large group of applications. In this section we will describe Fels and Olver moving frame together with its relation to the classical moving frame. We will also introduce some definitions useful to our study of geometric Poisson brackets and the relation to completely integrable systems of KdV type. From now on we will assume $M = G/H$ with G acting on M via left multiplication. We will also assume that curves in M are *parametrized* and, therefore, the group G does not act on the parameter.

DEFINITION 1.1. Let $J^k(\mathbb{R}, M)$ the k-jet space of curves, that is, the set of equivalence classes of curves in M up to k^{th} order of contact. If we denote by $u(x)$ a curve in M and by u_r the r derivative of u with respect to the parameter x, $u_r = \frac{d^r u}{dx^r}$, the jet space has local coordinates that can

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be represented by $u^{(k)} = (x, u, u_1, u_2, \dots, u_k)$. The group G acts naturally on parametrized curves, therefore it acts naturally on the jet space via the formula

$$g \cdot u^{(k)} = (x, g \cdot u, (g \cdot u)_1, (g \cdot u)_2, \dots)$$

where by $(g \cdot u)_k$ we mean the formula obtained when one differentiates $g \cdot u$ and then writes the result in terms of g, u, u_1, etc. This is usually called the *prolonged* action of G on $J^k(\mathbb{R}, M)$.

DEFINITION 1.2. A function

$$I : J^k(\mathbb{R}, M) \to \mathbb{R}$$

is called a kth order *differential invariant* if it is invariant with respect to the prolonged action of G.

DEFINITION 1.3. A map

$$\rho : J^k(\mathbb{R}, M) \to G$$

is called a left (resp. right) *moving frame* if it is equivariant with respect to the prolonged action of G on $J^k(\mathbb{R}, M)$ and the left (resp. right) action of G on itself.

The direct relation between classical moving frames and group-based moving frames is contained in the following theorem.

THEOREM 1.1. *([13]) Let $\Phi_g : G/H \to G/H$ be defined by multiplication by g. That is $\Phi_g([x]) = [gx]$. Let ρ be a group-based left moving frame with $\rho \cdot u = o$ where $o = [H] \in G/H$. Identify $d\Phi_\rho(o)$ with an element of $GL(n)$, where n is the dimension of M.*

Then, the matrix $d\Phi_\rho(o)$ contains in its columns a classical moving frame.

Intuitively, this Theorem illustrates how classical moving frames are described only by the action of the group on first order frames, while group-based moving frames are described by the complete action. Next we will introduce the equivalent to the classical Serre–Frenet equations. This concept if fundamental in our Poisson geometry study. Recall that the traditional road to finding a generating set of differential invariants for a curve is to differentiate the classical moving frame along the curve. The differential invariants are found as coefficients of these equations. Unfortunately this approach only works if the action of the group is affine. For example, in the case of $M = O(n+1, 1)/H$, the *Möbius sphere* or local model for flat conformal manifolds, classical Serre–Frenet equations fail to produce two differential invariants for curves ([6]), while in the case $M = \mathrm{PSL}(n)/H \cong \mathbb{R}\mathrm{P}^n$ the Serre–Frenet equations fail to produce *any* invariant at all. The missing invariants are those generated by the action of the group on higher order frames.

DEFINITION 1.4. Consider Kdx to be the horizontal component of the pullback of the left-invariant Maurer-Cartan form of the group G via a group-based left moving frame ρ. That is

$$K = \rho^{-1}\rho_x \in \mathfrak{g}$$

(K is the coefficient matrix of the equation satisfied by ρ). We call K the *Serre–Frenet equations* for the moving frame ρ.

The crucial property that makes the group-based picture useful to our study is the fact that, if the moving frame is properly chosen, a complete set of generating differential invariants can always be found among the coefficients of the Serre–Frenet equations. By a moving frame of minimal order we mean one for which $\rho \cdot u^{(k)}$ has as many constant coordinates as is allowed by the rank of the prolonged action on $\mathrm{J}^k(\mathbb{R}, M)$ (for example, if the rank is full, $\rho \cdot u^{(k)}$ should be constant). For more information on normalization constants, please see [4, 5].

THEOREM 1.2. *(A direct consequence from [4, 5]) Let ρ be a (left or right) moving frame of minimal order for a curve u. Then, the coefficients of the (left or right) Serre–Frenet equations for ρ contain a basis for the set of differential invariants of the curve. That is, any other differential invariant for the curve is a function of the entries of K and their derivatives with respect to x.*

From now on we will assume that G/H is flat and G is semisimple. Then ([20]) the Lie algebra $\mathfrak{g}$ has a gradation of length 1, that is

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{1.1}$$

with $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{h}$ is the Lie algebra of H. Therefore, locally G splits as $G_{-1} \cdot G_0 \cdot G_1$ with H given by $G_0 \cdot G_1$. The subgroup G_0 is called *the isotropic subgroup* of G and it is the component of G that acts linearly on G/H (for more information see [20]).

Choose a minimal order moving frame ρ with $\rho \cdot u = o$ and let $K = \rho^{-1}\rho_x$ be its Serre–Frenet equations. We split K according to the gradation (1.1), that is

$$K = K_1 + K_0 + K_{-1}.$$

It is known ([12]) that, if the moving frame is minimal, the term K_{-1} will always be constant.

DEFINITION 1.5. We call *differential invariants of projective type* those invariants that appear in the component K_1.

These invariants are those that are failed to be produced by the classical Serre–Frenet equations because they are generated by the action of the group on higher frames.

2. Geometric Poisson brackets. In this section we will introduce the definition of geometric Poisson bracket, that is, Hamiltonian structures

on the space of differential invariants of curves. For a complete description and proofs see [12, 13].

Consider the group of Loops $\mathcal{L}G = C^\infty(S^1, G)$ and its Lie algebra $\mathcal{L}\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$. One can define two natural Poisson brackets on $\mathcal{L}\mathfrak{g}^*$ (see [22] for more information), namely, if $\mathcal{H}, \mathcal{F} : \mathcal{L}\mathfrak{g}^* \to \mathbb{R}$ are two functionals defined on $\mathcal{L}\mathfrak{g}^*$ and if $L \in \mathcal{L}\mathfrak{g}^*$, we define

$$\{\mathcal{H}, \mathcal{F}\}_1(L) = \int_{S^1} \langle -\left(\frac{\delta \mathcal{H}}{\delta L}(L)\right)_x + ad^*(L)(\frac{\delta \mathcal{H}}{\delta L}(L)), \frac{\delta \mathcal{F}}{\delta L}(L)\rangle dx \qquad (2.1)$$

where $\langle , \rangle$ is the natural coupling between $\mathfrak{g}$ and $\mathfrak{g}^*$, and where $\frac{\delta \mathcal{H}}{\delta L}(L)$ is the variational derivative of $\mathcal{H}$ at L identified, as usual, with an element of $\mathcal{L}\mathfrak{g}$.

One also has a family of second brackets, namely

$$\{\mathcal{H}, \mathcal{F}\}_2(L) = \int_{S^1} \langle (ad^*(L_0)(\frac{\delta \mathcal{H}}{\delta L}(L)), \frac{\delta \mathcal{F}}{\delta L}(L)\rangle dx \qquad (2.2)$$

where $L_0 \in \mathfrak{g}^*$ is any constant element. Since $\mathfrak{g}$ is semisimple we can identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ and we will do so from now on.

The basis for the definition of a geometric Poisson bracket is contained in the following Theorem.

THEOREM 2.1. *([12]) Let $M = G/H$ be a flat semisimple manifold, with $\mathfrak{g}$ having a 1-gradation as in (1.1). Assume that for every curve in M we choose a minimal moving frame ρ with $\rho \cdot u = o$ and $\rho \cdot u_1 = \Lambda \in \mathfrak{g}_{-1}$ (we are identifying the tangent with $\mathfrak{g}_{-1}$). Then, locally around a given curve u, the space $\mathcal{K}$ of elements $K \in \mathcal{L}\mathfrak{g}$ defined by the Serre–Frenet equations for ρ can be described as a quotient $U/\mathcal{N}$, where U is an open set of $\Lambda \oplus \mathcal{L}\mathfrak{g}_0 \oplus \mathcal{L}\mathfrak{g}_1$ and where $\mathcal{N} = \mathcal{L}\mathcal{N}_0 \cdot \mathcal{L}G_1 \subset \mathcal{L}G_0 \cdot \mathcal{L}G_1$ acts on U via the Kac-Moody action*

$$a(n)(L) = n^{-1}n_x + n^{-1}Ln. \qquad (2.3)$$

The subgroup $\mathcal{N}_0$ is the isotropy subgroup of Λ in G_0.

Once we see that $\mathcal{K}$ (a space we can call the *space of differential invariants*) can be written as a quotient, the idea of Poisson reduction is unavoidable. Indeed, the *symplectic leaves* (or orbits where Hamiltonian flows lie) of bracket (2.1) are given by the orbits of $\mathcal{L}\mathfrak{g}^*$ under the action of $\mathcal{L}G$ defined by (2.3). Thus, one gets the following Theorem.

THEOREM 2.2. *([12]) The Poisson bracket (2.1) can be reduced to $\mathcal{K}$ so that there exists a well-defined Poisson bracket $\{,\}$ defined on a generating set of independent differential invariants. We call this bracket a geometric Poisson bracket.*

It is not true in general that (2.2) is also reducible to $\mathcal{K}$. In fact, one finds that for $M = \mathbb{R}\mathrm{P}^n$ and $G = \mathrm{PSL}(n+1)$ the second bracket (2.2) is also reducible to $\mathcal{K}$ when $L_0 = \Gamma^* \in \mathfrak{g}^*$, while if M is the so-called Lagrangian Grassmannian, $G = \mathrm{Sp}(4)$, the second bracket is never

reducible to $\mathcal{K}$ ([15]). A similar construction exists for affine geometries $M = G \ltimes \mathbb{R}^n/G$, G semisimple. Again, the first bracket is always reducible while the reduction of the second depends on the manifold. On the other hand, in all known cases, once we know that the second bracket is reducible, a biHamiltonian PDE (that is, a completely integrable system) exists for both reduced brackets. For example, in the case of $M = \mathbb{R}\mathrm{P}^n$ both brackets are reducible for $L_0 = \Gamma^*$ and the resulting brackets are the first and second Hamiltonian structures for generalized KdV equations. In the Euclidean space both structures can be reduced and the results are equivalent to the first and second Hamiltonian structures for the nonlinear Shrödinger equation. And so on. But there is more to it, one can also find a geometric evolution of curves of the form

$$u_t = F(u, u_1, u_2, \dots)$$

invariant under the group G, and such that the induced evolution on the generating system of differential invariants determined by $\mathcal{K}$ is the completely integrable system that was found using the geometric brackets. The precise relation between both equations is given by the following Theorem.

THEOREM 2.3. *([12]) Assume ρ is as in the statement of Theorem 2.1. Assume $u(t,x)$ is a solution of the evolution*

$$u_t = d\Phi_\rho(o)\mathbf{r} = r_1 V_1 + \cdots + r_n V_n \tag{2.4}$$

where $\mathbf{r} = (r_i)$ is a vector of differential invariants (recall that $d\Phi_\rho(o) = (V_1 \dots V_n)$ is a classical moving frame). Then, the evolution induced on the generating system of differential invariants defined by $\mathcal{K}$, call them $\mathbf{k}$, is Hamiltonian with associated Hamiltonian functional $h : \mathcal{K} \to \mathbb{R}$ whenever there exists an extension of h, $\mathcal{H} : U \to \mathbb{R}$, constant of the leaves of $\mathcal{N}$, such that $\left[\frac{\delta \mathcal{H}}{\delta L}(L)\right]_{-1} = \mathbf{r}$ (the subindex -1 indicates the component in K_{-1}).

Given a Hamiltonian system $\mathbf{k_t} = \xi_h(\mathbf{k})$ we will call its associated geometric evolution (2.4) *its geometric realization in M*. Thus, it is known that the Vortex filament evolution is a Euclidean geometric realization of the nonlinear Shrödinger equation, as it was shown by Hasimoto in [9]. The Schwarzian KdV equation is a projective geometric realization of the KdV equation, and so on.

A completely integrable PDE could, in principle, have more than one geometric realization. Indeed, modified KdV systems have geometric realizations in several manifolds (see [11], [18], [23], [1]) but all these manifolds share a common Riemannian structure that has a fundamental role in the generation of the Hamiltonian structures and of their associated geometric realizations. Furthermore, the known results indicate that the reduction of the second bracket (2.2) points at the existence of a completely integrable PDE induced by geometric realizations, but the geometric character of the manifold determines what type of PDE can have a geometric realization in the manifold. Indeed, the Sowada-Koterra equation has a geometric realization in the equi-affine case $M = SL(2) \ltimes \mathbb{R}^2/SL(2)$ ([19], [13]) while

the KdV equation has no such realization, it has a projective one instead. But both equi-affine and projective *bi-Hamiltonian geometric brackets are the same*. The Hamiltonian structures are known to be the same for both Sowada–Koterra and KdV, but the affine character of the equi-affine manifold allows for the existence of a geometric realization of Sowada–Koterra, and not for KdV, while the projective character of $\mathbb{R}\mathrm{P}^1$ allows for the existence of a geometric realization for KdV but not for Sowada–Koterra.

3. Completely integrable systems of KdV type associated to differential invariants of projective type. Finally, in our last section we would like to drive home the last point of our previous section by describing how general projective-type differential invariants have associated integrable PDEs, in the sense of our previous section, always of KdV type. They also have geometric realizations in G/H, assuming we restrict our initial curves to curves whose non-projective differential invariants vanish. The following Theorem was conjecture in [14].

THEOREM 3.1. *The geometric Poisson bracket associated to curves on G/H can be further restricted to the submanifold of projective differential invariants, that is, the submanifold of $\mathcal{K}$ for which all non-projective differential invariants vanish. The resulting Poisson structures and the behavior of the invariants of projective type under properly chosen geometric flows are either* projective, conformal or Lagrangian like*, always of KdV-type.*

In what remains of this paper we will describe the three types of behaviour, projective, conformal and Lagrangian. We will also describe the progress made in proving this Theorem.

3.1. The Lagrangian case. Let $G = \mathrm{Sp}(2n)$. If $g \in G$ then, locally

$$g = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & \Theta^{-T} \end{pmatrix} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}$$

with u and S symmetric and $\Theta \in \mathrm{GL}(n)$. This factorization corresponds to the splitting given by the gradation (1.1). The subgroup H is locally defined by matrices of the form

$$\begin{pmatrix} \Theta & 0 \\ 0 & \Theta^{-T} \end{pmatrix} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix}.$$

The manifold G/H is usually called the n dimensional *Lagrangian Grassmanian* and it is identified with the manifold of Lagrangian planes in $\mathbb{R}^{2n}$. The following Theorem describes the manifold $\mathcal{K}$ for curves of Lagrangian planes in $\mathbb{R}^{2n}$ under the action of $\mathrm{Sp}(2n)$.

THEOREM 3.2. *([15]) There exists a minimal order moving frame ρ along a curve of Lagrangian planes such that its Serre–Frenet equations are given by*

$$K = \rho^{-1}\rho_x = \begin{pmatrix} K_0 & I \\ K_1 & K_0 \end{pmatrix}$$

where K_0 is skew-symmetric and contains all differential invariants of order 4, and where $K_1 = -\frac{1}{2}\mathcal{S}_d$ with $\mathcal{S}_d$ diagonal and containing along the diagonal the eigenvalues of the Lagrangian Schwarzian derivative *(Ovsienko [21])*

$$\mathcal{S}(u) = u_1^{-1/2}\left(u_3 - \frac{3}{2}u_2u_1^{-1}u_2\right)u_1^{-1/2}.$$

The entries of K_0 and K_1 are functionally independent differential invariants for curves of Lagrangian planes in $\mathbb{R}^{2n}$ under the action of $\mathrm{Sp}(2n)$ *generating all other differential invariants. The n differential invariants that appear in K_1 are the invariants of projective type.*

Now we describe some of the geometric flows that preserve the value of K_0. Therefore, geometric flows as below will affect only invariants of projective type.

THEOREM 3.3. *([15]) Assume $u : J \subset \mathbb{R}^2 \to \mathrm{Sp}(2n)/H$ is a flow solution of*

$$u_t = \Theta^T u_1^{1/2}\ \mathbf{h}\ u_1^{1/2}\Theta \tag{3.1}$$

where $\Theta(x,t) \in O(n)$ diagonalizes $\mathcal{S}(u)$ and where $\mathbf{h}$ is a symmetric matrix of differential invariants. Assume $\mathbf{h}$ is diagonal. Then the flow preserves K_0.

Finally, the integrable PDEs associated to geometric flows of Lagrangian planes are given in our next Theorem.

THEOREM 3.4. *([15]) Let $\mathcal{K}_1$ be the submanifold of $\mathcal{K}$ given by $K_0 = 0$. Then, the geometric bracket restricts to $\mathcal{K}_1$ to induce a decoupled system of n second Hamiltonian structures for KdV. Bracket (2.2) also reduces to $\mathcal{K}_1$ (even though it does not in general reduce to $\mathcal{K}$ for any value of L_0). The reduction for a special value of $L_0 \in g_1$ is a decoupled system of n first KdV Hamitonian structures.*

Furthermore, assume $u(t,x)$ is a flow solution of (3.1) and assume $u(0,x)$ is an initial condition for which $K_0 = 0$. Then the differential invariants $\mathcal{S}_d$ of the flow satisfies the equation

$$(\mathcal{S}_d)_t = \left(\mathbf{D}^3 + \mathcal{S}_d\mathbf{D} + (\mathcal{S}_d)_x\right)\mathbf{h}$$

where $\mathbf{D}$ is the diagonal matrix with $\frac{d}{dx}$ down its diagonal.

From this Theorem, it is clear that if $\mathbf{h} = \mathcal{S}_d$, then $\mathcal{S}_d$ is the solution of a *decoupled system of n KdV equations*

$$(\mathcal{S}_d)_t = (\mathcal{S}_d)_{xxx} + 3\mathcal{S}_d(\mathcal{S}_d)_x.$$

This would also be the situation for curves in another different case, that of the manifold $M = O(2n,2n)/H$. In this particular case it was shown in [14] that the differential invariants of projective type are represented in $\mathcal{K}$ by skew-symmetric matrices that diagonalise in off-diagonal blocks. The geometric Poisson brackets behave in exactly the same way as those in the Lagrangian Grassmannian manifold. For more information see [14].

3.2. Conformal case. Consider $G = O(n+1,1)$. If $g \in G$ then, locally,

$$g = \begin{pmatrix} 1 & u^T & \frac{1}{2}u^T u \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ v & I & 0 \\ \frac{1}{2}v^T v & v^T & 1 \end{pmatrix}$$

with $A \in O(n)$, $u, v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Again, the above splitting corresponds to the gradation (1.1) and H is given by matrices with factorization

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ v & I & 0 \\ \frac{1}{2}v^T v & v^T & 1 \end{pmatrix}.$$

The manifold $O(n+1,1)/H$ is *the Möbius sphere*, the local model for flat conformal manifolds. Differential invariants for conformal curves were first found in [6] and its group-based representation was described in [16]. The following Theorem describes the Serre-Frenet equations associated to minimal moving frames.

THEOREM 3.5. *([15]) There exists a minimal order (left) moving frame ρ along a curve in the Möbius sphere with $\rho \cdot u = o, \rho \cdot u_1 = e_1$ and such that its Serret-Frenet equations are given by*

$$K = \begin{pmatrix} 0 & e_1^T & 0 \\ k_1 e_1 + k_2 e_2 & K_0 & e_1 \\ 0 & k_1 e_1^T + k_2 e_2^T & 0 \end{pmatrix}$$

where k_1 and k_2 are third order conformal invariants of projective type *and where K_0 are fourth order and higher. The vectors e_i represent the standard basis in $\mathbb{R}^n$.*

In this case there are only two generating differential invariants of projective type, namely k_1 and k_2. Again, the behavior of the Poisson brackets (2.1) and (2.2) with respect to this submanifold is spotless.

THEOREM 3.6. *([15]) Let $\mathcal{K}_1$ be the submanifold of $\mathcal{K}$ given by $K_0 = 0$. Then, the geometric bracket restricts to $\mathcal{K}_1$ to induce the second Hamiltonian structure for a complexly coupled KdV system. Bracket (2.2) also reduced to $\mathcal{K}_1$ to produce the first Hamiltonian structure for this system.* And, again, the geometric realization for complexly coupled KdV is found.

THEOREM 3.7. *([14]) Assume $u : J \subset \mathbb{R}^2 \to O(n+1,1)/H$ is a solution of*

$$u_t = h_1 T + h_2 N$$

where T and N are conformal tangent and normal (see [16]). Then the flow preserves K_0.

If $K_0 \to 0$, the evolution of k_1 and k_2 becomes

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}D^3 + k_1 D + D k_1 & k_2 D + D k_2 \\ k_2 D + D k_2 & \frac{1}{2}D^3 - k_1 D - D k_1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

In particular, if $h_1 = k_1$ and $h_2 = k_2$, then k_1 and k_2 are solutions of a *complexly coupled system of KdV equations*.

As it was shown in [14], a very similar situation holds true for the case $G = O(p+q, q)$. In this case one also finds only two differential invariants of projective type and the same reductions and similar geometric realizations can be found.

Finally, our last case is the projective one.

3.3. Projective case. Let $G = \mathrm{PSL}(n+1)$. If $g \in G$ then, locally

$$g = \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ v^T & 1 \end{pmatrix}$$

with $u, v \in \mathbb{R}^n$ and $A \in \mathrm{GL}(n)$.

The manifold $\mathrm{PSL}(n+1)/H$ can be identified with the n *projective space* $\mathbb{R}\mathrm{P}^n$. We will simply state the results in this case.

THEOREM 3.8. *(reformulation of Wilczynski [24]) There exists a minimal order moving frame ρ along a curve in $\mathbb{R}\mathrm{P}^n$ such that its Serret-Frenet equations are defined by matrices of the form*

$$K = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ k_1 & k_2 & \dots & k_n & 0 \end{pmatrix}$$

where k_i, $i = 1, \dots, n$ are in general a combination of the Wilczynski projective invariants.

THEOREM 3.9. *([12]) The n-projective geometric Poisson bracket is given by the Adler–Gel'fand–Dikii bracket or second Hamiltonian structure for KdV. The bracket (2.2) reduces for $L_0 = e_n^*$ and its reduction is the first Hamiltonian structure for generalised KdV equations.*

THEOREM 3.10. *([17]) Assume $u : J \subset \mathbb{R}^2 \to PSL(n)/H$ is a solution of*

$$u_t = h_1 T_1 + h_2 T_2 + \cdots + h_n T_n$$

where T_i form a projective classical moving frame.

Then, $\mathbf{k} = (k_i)$ satisfy an equation of the form

$$\mathbf{k}_t = P\mathbf{h}$$

where $\mathbf{k} = (k_1, \dots, k_n)^T$, $\mathbf{h} = (h_1, \dots, h_n)^T$ and where P is the Poisson tensor defining the Adler–Gel'fand–Dikii Hamiltonian structure. *In particular, if $\mathbf{h} = \mathbf{k}$ then $\mathbf{k}$ satisfies a* generalized KdV system *of equations.*

These are the three cases that seem to describe the pattern for other cases. In [10] Kobayashi showed how the action of the group G on a

homogeneous space G/H with a gradation as in (1.1) decouples on actions by its simple terms. The classification of all the simple cases was found in [10]. They correspond to all cases mentioned in this paper plus $G = O(2n+1, 2n+1), \mathrm{PSL}(p+q)$ and two with exceptional Lie algebras. These four cases are still open, although the conjecture is that $O(2n+1, 2n+1)$ might behave as a Lagrangian type while $\mathrm{PSL}(p+q)$ is probably projective type. The principal obstacle in most of these studies is finding minimal moving frames and the classification of differential invariants of projective type in a simple enough way that the resulting integrable equations can be identified. The moving frames and differential invariants are unknown for the remaining cases and need to be found as a previous step to the study of their geometric brackets.

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