

GEOMETRIC POISSON BRACKETS ON GRASSMANNIANS AND CONFORMAL SPHERES

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ABSTRACT. In this paper we relate the geometric Poisson brackets on the 2-Grassmannian in \mathbb{R}^4 and on the $(2,2)$ Möbius sphere. We show that, when written in terms of local moving frames, the geometric Poisson bracket on the Möbius sphere does not restrict to the space of differential invariants of Schwarzian type. But when the concept of conformal natural frame is transported from the conformal sphere into the Grassmannian, and the Poisson bracket is written in terms of the Grassmannian natural frame, it restricts and results into either a decoupled system or a complexly coupled system of KdV equations, depending on the character of the invariants. We also show that the biHamiltonian Grassmannian geometric brackets are equivalent to the non-commutative KdV biHamiltonian structure. Both integrable systems and Hamiltonian structure can be brought back to the conformal sphere.

1. INTRODUCTION

Given a flat homogeneous space, one can define a Hamiltonian structure on the space of differential invariants (curvatures) of parametrized curves ([12]). These structures are often linked to completely integrable PDEs and to their geometric realizations (invariant curve flows on the homogeneous space inducing the integrable system on its invariants). There has recently been a flurry of literature studying the existence of these integrable systems and their associated geometric flows, for example see [1], [3], [8], [9], [10], [17], [19], [20] and references within.

We will say a differential invariant I is of Schwarzian type whenever $\phi^*I = ((\phi')^2I) \circ \phi^{-1} + S(\phi) \circ \phi^{-1}$, where ϕ^*I represents the pull-back of I by a diffeomorphism ϕ and where $S(\phi)$ indicates the Schwarzian derivative of ϕ (the Schwarzian derivative itself behaves this way under reparametrization). In [14] the author conjectured that the nature of the geometry (and its invariants) was linked to the type of integrable systems they could realize. In particular, she conjectured that the existence of differential invariants of projective (or Schwarzian) type would result in geometric realizations of equations of KdV type (as it appeared in [14] and [15], for example), while the existence of invariants of Riemannian type would result in geometric realizations of nonlinear Schrödinger equations (NLS), modified KdV (mKdV) and Sine Gordon (as in [1], [17] and [20], for example). In particular, one can obtain a geometric realization of the KdV equation by flows in \mathbb{RP}^1 , of generalized KdV equations by flows on \mathbb{RP}^n , of a system of complexly coupled KdV

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equations by conformal flows and of a decoupled system of KdV equations by a flow in the Lagrangian Grassmannian. Eastwood conjectured that this dichotomy might be related to the existence of preferred parametrizations, projective versus affine, as in [4]. Many of these geometric realizations are given by flows for which all non-Schwarzian invariants vanish or are constant (i.e. initial curves are restricted). That is, they are in fact completely integrable level sets of associated curve flows which are not completely integrable themselves. More interestingly, the existence of these integrable level sets is always linked to the reduction of a bi-Hamiltonian Poisson structure to the submanifold of vanishing non-Schwarzian invariants (as in [14] and [15]).

Most of the above examples are particular cases of flat parabolic geometries, that is, homogenous spaces of the form G/P , G semisimple and P a parabolic subgroup. (In fact, they are instances of parabolic geometries associated to $|1|$ -gradings of the algebra.) One such case does not seem to behave the way the other cases do, namely the spinor case $G = O(2n, 2n)$. The author showed in [14] that, even though spinor curves do possess differential invariants of Schwarzian type, the geometric Poisson structure associated to flows of spinors does not reduce to the submanifold of vanishing non-Schwarzian invariants. Furthermore, the somehow expected flow possessing a decoupled system of KdV equations as level set does not preserve this submanifold so that one cannot find a geometric realization of a system of decoupled KdV by flows of spinors.

In this paper we study the flat Grassmannian case of 2 dimensional planes in \mathbb{R}^4 . This homogeneous space can be identified with the homogeneous manifold $SL(2+2, \mathbb{R})/P$, for a properly chosen parabolic subgroup P . The notation $SL(2+2)$ refers to the action of $SL(4)$ on the manifold, as shown in Section 3. Since the manifold is flat, the Cartan connection of the manifold will be given by the Maurer-Cartan form (our results are local). As we will see, the group $SL(2+2, \mathbb{R})$ is a double cover of $O(3, 3)$ and the cover induces an equivalency of parabolic geometries. Indeed, the oriented conformal sphere $SO(3, 3)/P$ can also be viewed as the spin manifold $Spin(3, 3)/P$, itself isomorphic to $SL(2+2)/P$. At the infinitesimal level the isomorphism is given by an isomorphism of the Lie algebras and their associated gradations. Our original intention was to translate our knowledge of the conformal case into the Grassmannian. As it turned out, we also ended up learning more about the conformal case from the Grassmannian situation.

Moving frames and differential invariants for curves in Grassmannian manifolds $Gr(p, q) = SL(p+q)/P$ where P is a suitable chosen parabolic subgroup, are not well-known in general. For the case $Gr(nr, r)$, a special type of non-local invariants were found in [18]. These invariants correspond to a Laguerre-Forsyth canonical form for the Serret-Frenet equations, and we will use them at the end of the paper. In section 3 we will find a local moving frame along curves in $Gr(2, 2)$ and we will find the differential invariants they generate. We will show that two of the four generating invariants are invariants of Schwarzian type. In Theorem 4 we will find explicitly the most general form of an invariant Grassmannian flow and we will show that even those who have normalized coefficients do not preserve the submanifold of vanishing non-Schwarzian invariants; when these invariants vanish, the invariant flow blows up. This will imply that the geometric Poisson bracket does not restrict to the space where the non-Schwarzian invariants vanish, much like the situation in the spinor case [14]. This seems to be somehow counterintuitive, integrable level

sets do exist in the conformal sphere of signature $(n, 0)$ for which we can find a complexly coupled system of KdV equations.

We show that the Grassmannian problem lies in the choice of moving frame. For this we notice that a *local* (i.e. depending on the curves and its derivatives) choice of moving frame in the conformal sphere results in the same type of problem the Grassmannian case had. On the other hand, if we choose a *natural moving frame*, a generalization of the non-local natural Euclidean frame, then the level set is preserved and both Hamiltonian structures can be reduced. In section 5 we define natural frames for both the conformal sphere of signature $(2, 2)$ and for Grassmannian curves. We then prove that we can find a Grassmannian geometric realization inducing a complexly coupled system of KdV equations on the Grassmannian curvatures of projective type. Furthermore, we show that there is also a geometric realization of a decoupled system of two KdV equations.

Finally, we show that, when written in terms of the moving frame generating the non-local invariants appearing in [18], the biHamiltonian geometric structure *on the complete Grassmannian* is equal to the non-commutative KdV biHamiltonian structure. We also show that the noncommutative KdV equation has a Grassmannian geometric realization. The noncommutative KdV equation and its biHamiltonian structures were defined in [16]. Given the relation to the conformal sphere, these produce also conformal biHamiltonian structures and a geometric realization for this system. The only conformal realizations that were previously known were those of the coupled KdV system. Using the isomorphism, we also prove that the Poisson brackets on the conformal $(2, 2)$ sphere are the noncommutative KdV structures.

2. GRASSMANNIAN-CONFORMAL PARABOLIC EQUIVALENCE

2.1. Description of the manifolds. Let us first realize $\text{Gr}_2(\mathbb{R}^4)$ as the homogeneous space $\text{SL}(4, \mathbb{R})/P_G$ where P_G is the parabolic subgroup of $\text{SL}(4, \mathbb{R})$ defined by matrices of the form

$$\begin{pmatrix} A & \mathbf{0} \\ C & B \end{pmatrix}$$

where $A, B, C, \mathbf{0} \in M_{2 \times 2}$ and where $\det A \det B = 1$. The subindex G in P_G indicates its association to the Grassmannian. Its Lie algebra \mathfrak{p}_G is defined by similarly shaped matrices with vanishing trace. This quotient corresponds to a gradation of the algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, where $\mathfrak{p}_G = \mathfrak{g}_1 \oplus \mathfrak{g}_0$, and \mathfrak{g}_{-1} is defined by the upper right block (the dual to \mathfrak{g}_1).

We now describe the conformal sphere with signature $(2, 2)$. Let $J \in M_{6 \times 6}$ be the matrix

$$(2.1) \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can realize the group $\text{SO}(3, 3)$ as the identity component of the group $\text{O}(3, 3)$ defined as

$$\text{O}(3, 3) = \{A \in \text{GL}(6, \mathbb{R}), \text{ such that } A^t J A = J\}.$$

With this realization the Lie algebra will be given by matrices which are skew symmetric with respect to the secondary diagonal, that is, $X \in \mathfrak{so}(3, 3)$ whenever $X^t J + JX = 0$

$$(2.2) \quad X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 0 \\ x_{21} & x_{22} & x_{23} & x_{24} & 0 & -x_{15} \\ x_{31} & x_{32} & x_{33} & 0 & -x_{24} & -x_{14} \\ x_{41} & x_{42} & 0 & -x_{33} & -x_{23} & -x_{13} \\ x_{51} & 0 & -x_{42} & -x_{32} & -x_{22} & -x_{12} \\ 0 & -x_{51} & -x_{41} & -x_{31} & -x_{21} & -x_{11} \end{pmatrix}.$$

Next, define P_C (the subindex C indicates its association with the conformal sphere) to be the parabolic subgroup of $\mathrm{SO}(3, 3)$ given by the stabilizer of the line

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ * \end{pmatrix} \in \mathbb{R}^6$$

under the linear action of $\mathrm{SO}(3, 3)$; that is, the stabilizer of the basepoint in \mathbb{RP}^5 under the projective action. Its orbit is the non-singular quadric

$$Q \equiv \{[v] \in \mathbb{RP}^5, \text{ such that } v^t J v = 0\} = \mathrm{SO}(3, 3)/P_C.$$

The parabolic Lie algebra \mathfrak{p}_C is given by those elements in $\mathfrak{so}(3, 3)$, as in (2.2), for which $x_{1i} = 0$, $i = 2, 3, 4, 5$. As before, the quotient is related to a gradation of the algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ with \mathfrak{g}_{-1} dual to \mathfrak{g}_1 and $\mathfrak{p}_C = \mathfrak{g}_1 \oplus \mathfrak{g}_0$.

2.2. Isomorphism between the homogeneous manifolds. It is well-known that there exists an isomorphism of homogeneous spaces

$$\mathrm{Gr}_2(\mathbb{R}^4) \cong Q.$$

The isomorphism is induced by a homomorphism at the Lie group level. Specifically, for $A \in \mathrm{SL}(4, \mathbb{R})$, define $\Phi(A) \in \mathrm{Hom}(\Lambda^2 \mathbb{R}^4, \Lambda^2 \mathbb{R}^4)$ by the usual induced action on simple vectors $v \wedge w$. That is

$$\Phi(A)(v \wedge w) = Av \wedge Aw.$$

We can identify $\Lambda^2 \mathbb{R}^4$ with \mathbb{R}^6 through the choice of basis

$$\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_4 \wedge e_2, e_3 \wedge e_4\}.$$

Under this representation, $\Phi(A)$ is defined by an $\mathrm{SO}(3, 3)$ matrix. Through straightforward calculations we see that, if $A, D \in M_{2 \times 2}$, $B, C \in M_{2 \times 2}$ with $\det B \det C = 1$, then

$$(2.3) \quad \Phi\left(\begin{pmatrix} I & A \\ \mathbf{0} & I \end{pmatrix}\right) = \begin{pmatrix} 1 & a_{21} & a_{22} & -a_{11} & a_{12} & \det A \\ 0 & 1 & 0 & 0 & 0 & -a_{12} \\ 0 & 0 & 1 & 0 & 0 & a_{11} \\ 0 & 0 & 0 & 1 & 0 & -a_{22} \\ 0 & 0 & 0 & 0 & 1 & -a_{21} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(2.4) \quad \Phi\left(\begin{pmatrix} I & \mathbf{0} \\ D & I \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ d_{12} & 1 & 0 & 0 & 0 & 0 \\ d_{22} & 0 & 1 & 0 & 0 & 0 \\ -d_{11} & 0 & 0 & 1 & 0 & 0 \\ d_{21} & 0 & 0 & 0 & 1 & 0 \\ \det D & -d_{21} & d_{11} & -d_{22} & -d_{12} & 1 \end{pmatrix}$$

$$(2.5) \quad \Phi\left(\begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{pmatrix}\right) = \begin{pmatrix} \det B & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{11}c_{11} & b_{11}c_{12} & b_{12}c_{11} & -b_{12}c_{12} & 0 \\ 0 & b_{11}c_{21} & b_{11}c_{22} & b_{12}c_{21} & -b_{12}c_{22} & 0 \\ 0 & b_{21}c_{11} & b_{21}c_{12} & b_{22}c_{11} & -b_{22}c_{12} & 0 \\ 0 & -b_{21}c_{21} & -b_{21}c_{22} & -b_{22}c_{21} & b_{22}c_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \det C \end{pmatrix}.$$

The map Φ is a double cover of $\mathrm{SO}(3, 3)$ by $\mathrm{SL}(4, \mathbb{R})$, which also maps P_G into P_C . Notice that Φ is a double cover on the parabolic subgroups, while it is one-to-one between the sections of $\mathrm{SL}(4, \mathbb{R})/P_G$ and $\mathrm{SO}(3, 3)/P_C$ defined by (2.3). Therefore, the map induces the desired isomorphism between homogeneous spaces. Clearly Φ induces a graded map at the Lie algebra level.

3. THE LOCAL GEOMETRY OF GRASSMANNIAN CURVES

Let $\mathrm{SL}(p+p) \subset \mathrm{GL}(p+p)$ be the simple linear group acting on $\mathfrak{gl}(p)$ according to the action of $\mathrm{SL}(2p)$ on the homogeneous space $M = \mathrm{SL}(2p)/H$, where $H \subset \mathrm{SL}(2p)$ are matrices of the form

$$\begin{pmatrix} E & 0 \\ C & D \end{pmatrix}$$

with $E, C, D \in M_{p \times p}$. Assume we are in a neighborhood of the identity and so we can locally factor an element $g \in \mathrm{SL}(2p)$ into the product

$$g = \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}.$$

Then, a representation for the homogeneous manifold M is given by the section defined by matrices of the form

$$\begin{pmatrix} I & u \\ 0 & I \end{pmatrix}.$$

Using this section we can write the $\mathrm{SL}(p+p)$ action on $\mathfrak{gl}(p)$ as determined by the relation

$$g \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & g \cdot u \\ 0 & I \end{pmatrix} h$$

where $h \in H$. This relation determines the action uniquely to be

$$(3.1) \quad g \cdot u = A(u + Y)(B + ZA(u + Y))^{-1}.$$

3.1. Group-based moving frames for Grassmannian generic curves. In this section we will use the normalization method described in [5] to find a group-based moving frame along generic parametrized curves in the manifold $\mathrm{Gr}_2(\mathbb{R}^4)$. Let $J^{(k)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$ be the k -jet space of curves in $\mathrm{Gr}_2(\mathbb{R}^4)$, i.e., the set of equivalence classes of curves in $\mathrm{Gr}_2(\mathbb{R}^4)$ up to k -contact order. An m -order left (resp. right) group-based moving frame is a map

$$\rho : J^{(m)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4)) \rightarrow \mathrm{SL}(2p)$$

equivariant with respect to the prolonged action of $\mathrm{SL}(2p)$ on $J^{(k)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$ and the left (resp. right) action of $\mathrm{SL}(2p)$ on itself. Let $u_r = \frac{d^r u}{dx^r}$, where x is the parameter. In the case at hand, the prolonged action is defined by the relation

$$g \cdot u_r = (g \cdot u)_r$$

where the right hand side represents the formula given by differentiating r times the action $g \cdot u$, and writing it in terms of x, u, u_1, \dots, u_r , the coordinates in $J^{(k)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$, $r \leq k$.

In order to find a left moving frame ρ along a generic curve $u(x)$, we will normalize the prolonged action of $\mathrm{SL}(2p)$ on $J^{(m)}(\mathbb{R}, \mathrm{Gr}_2(\mathbb{R}^4))$, up to a certain order m . The choice of normalization (which defines a cross-section on the prolonged orbits of the group) is not unique and can be arbitrary as far as we retain full rank. If ρ is a left (resp. right) moving frame, we call $K = \rho^{-1} \rho_x$ (resp. $K = \rho_x \rho^{-1}$) its associated *Maurer-Cartan matrix*. A theorem by Hubert ([7]) states that, if ρ is found via a normalization process, the entries of K and their derivatives functionally generate all other differential invariants of the curve. Different choices of normalization will give rise to different Maurer-Cartan matrices and different invariants. Our particular choices are made seeking both simplicity and a direct relation between the invariants (to this end the normalization constants are coordinated in both examples). Simplicity is important, a complicated Maurer-Cartan matrix will result in a difficult Hamiltonian study.

At each step we will normalize fully, i.e., we will normalize as many terms as permitted by the rank of the action. The terms that cannot be normalized will be differential invariants of the action. This process will determine an element g completely in terms of u and its derivatives. It is known (see [5]) that $\rho^{-1} = g$ found through this process is a right moving frame. A left moving frame is given by its inverse ρ .

We proceed to determine the (right) frame for the case at hand.

Zeroth normalization equation. For first normalization constant we will choose $i_0 = 0$. The first normalization equations will be equations of zero differential order

$$g \cdot u = A(u + Y) (B + ZA(u + Y))^{-1} = i_0 = 0$$

which is satisfied by the choice $Y = -u$. We have no zero order differential invariants.

First normalization equation. The next equations are the first order normalization equations $g \cdot u_1 = i_1$. We will make the normalization choice $i_1 = I$. After substituting the previous normalization choice ($u + Y = 0$), the equation becomes

$$(3.2) \quad \begin{aligned} g \cdot u_1 &= Au_1 (B + ZA(u + Y))^{-1} \\ -A(u + Y) (B + ZA(u + Y))^{-1} Z (B + ZA(u + Y))^{-1} &= Au_1 B^{-1} = i_1 = I. \end{aligned}$$

This equation is satisfied with the choice

$$A = Bu_1^{-1}.$$

Since $g \in \mathrm{SL}(2p)$, we also have $\det A \det B = 1$ and so

$$\det B = (\det u_1)^{1/2}.$$

As expected, we have no first order differential invariants. Let us call $F = (B + ZA(u + Y))$.

Second normalization equation. After differentiating again, the second order normalizations are found by substituting previous normalization values (in this

case $u + Y = 0$ and $A = Bu_1^{-1}$) and making the result equal to a constant i_2 . In this case we choose $i_2 = 0$. That is

$$(3.3) \quad \begin{aligned} g \cdot u_2 &= Au_2F^{-1} - 2Au_1F^{-1}ZAu_1F^{-1} \\ &- A(u + Y)F^{-1}Z(Au_2F^{-1} - 2Au_1F^{-1}ZAu_1F^{-1}) \\ &= Bu_1^{-1}u_2B^{-1} - 2Z = i_2 = 0. \end{aligned}$$

This is solved with the choice

$$Z = \frac{1}{2}Bu_1^{-1}u_2B^{-1}$$

and we have no second order invariants. At this point we only have left the determination of B (although not its determinant).

Third normalization equation. The third order normalization equations are, after some simplification, given by

$$(3.4) \quad g \cdot u_3 = B \left(u_1^{-1}u_3 - \frac{3}{2}u_1^{-1}u_2u_1^{-1}u_2 \right) B^{-1} = i_3.$$

Definition 1. We call

$$S(u) = u_1^{-1}u_3 - \frac{3}{2}u_1^{-1}u_2u_1^{-1}u_2$$

the *Schwarzian derivative of the Grassmannian curve u* .

Equation $BS(u)B^{-1} = i_3$ does not have full rank on B for any choice of i_3 , we can at most reduce $S(u)$ to a certain normal form under conjugation. Let us assume now that $p = 2$. The action $U \rightarrow BUB^{-1}$ has two invariants, namely, the determinant and trace of U . That means that the rank of this action is two and we will be able to use at most two third order normalization equations. Therefore, there will be two third order differential invariants given by the entries that cannot be further normalized. Notice that these are Grassmannian invariants of Schwarzian-type.

The following result is a consequence of theorems that can be found, for example, in [6].

Proposition 1. *Two generating and (functionally) independent third order differential invariants for a Grassmannian curve u in M are given by the determinant and the trace of its Schwarzian derivative. That is, any other third order differential invariant of u must be a function of these two.*

There are many possible choices for i_3 . Our choice will be to normalize i_3 depending on the nature of its eigenvalues (real or complex). Let us write

$$S(u) = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in that case i_3 is given by

$$\begin{aligned} i_3 &= BS(u)B^{-1} \\ &= (\det u_1)^{-1/2} \begin{pmatrix} d(as_1 + bs_3) - c(as_2 + bs_4) & -b(as_1 + bs_3) + a(as_2 + bs_4) \\ d(cs_1 + ds_3) - c(cs_2 + ds_4) & -b(cs_1 + ds_3) + a(cs_2 + ds_4) \end{pmatrix}. \end{aligned}$$

If we call $a/b = \alpha$ and $c/d = \beta$.

In the generic case of real eigenvalues, diagonalizing i_3 is equivalent to solving the same equation for both α and β , namely

$$(3.5) \quad P(\alpha) = \alpha^2s_2 + \alpha(s_4 - s_1) - s_3 = 0, \quad P(\beta) = \beta^2s_2 + \beta(s_4 - s_1) - s_3 = 0.$$

Since $\det B \neq 0$ we need this equation to have two different solutions and we need α and β to be those two different solutions (the discriminant condition on this equation is the same as the one for the characteristic equation for $S(u)$). Therefore

$$i_3 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

In the generic case of complex eigenvalues, the normal form will be

$$(3.6) \quad \begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}$$

and the corresponding conditions to achieve it are

$$(3.7) \quad 2\alpha\beta s_2 + (\beta + \alpha)(s_4 - s_1) - 2s_3 = 0, \quad b^2 P(\alpha) = d^2 P(\beta),$$

where $P(\alpha)$ and $P(\beta)$ are given as in (3.5).

In both cases, the condition

$$(3.8) \quad (\alpha - \beta)bd = \det B = (\det u_1)^{1/2}$$

together with the two equations solve for a, c and d in terms of b , in the real case, or for α, b and d in terms of β in the complex case. The real case is straightforward, the complex case might require a little more explanation. From (3.7) we obtain

$$(3.9) \quad \alpha = \frac{\beta(s_4 - s_1) - 2s_3}{s_1 - s_4 - 2\beta s_2}$$

and

$$(3.10) \quad b^4 = \frac{1}{4} \frac{(2\beta s_2 + s_4 - s_1)^2 \det u_1}{P(\alpha)P(\beta)}.$$

Using (3.9) one can see that

$$(3.11) \quad P(\alpha) = -\frac{P(\beta)}{(2\beta s_2 + s_4 - s_1)^2} \Delta$$

where $\Delta = (s_4 - s_1)^2 + 4s_3 s_2$ is the discriminant of the characteristic polynomial of $S(u)$, and hence negative in our case. Therefore

$$b^4 = \frac{1}{4} \frac{-\Delta(2\beta s_2 + s_4 - s_1)^4 \det u_1}{P(\beta)^2}.$$

From (3.7) and (3.11), we also get

$$(3.12) \quad d^4 = -\frac{1}{4} \frac{\Delta^3 \det u_1}{P(\beta)^2}.$$

Thus, the only condition we need to be able to solve for b and d is $\det u_1 > 0$, which has been required early on, in view of (3.8).

Fourth normalization equation. Finally, if we write the fourth order normalization equations and we use previous normalization we will obtain an equation also of the form

$$(3.13) \quad BR(u)B^{-1} = i_4,$$

for some fourth order matrix $R(u)$ involving derivatives of u , but not B . We do not really need here its explicit expression, but we need to choose the last normalization equation carefully so as to simplify later calculations.

Real case.

Assume first we are in the real case. If we denote $R(u)$ by

$$R(u) = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}$$

then normalizing its (1,2) entry, for example, will give us the remaining missing equation. Although one usually chooses constants to normalize, we can also choose expressions that depend on invariants as we simply need a section of the prolonged orbit (see [5]). The resulting generation of invariants will remain unchanged, although the generators will be different. Assume we have diagonalized the Schwarzian derivative and it is given by

$$\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Generically $k_1 \neq k_2$ and we can assume that $k_1 - k_2$ will not vanish near the basepoint. Let us choose as last normalization equation to be the (1,2) entry equal $k_2 - k_1$

$$(3.14) \quad (\det u_1)^{-1/2} b^2 (\alpha^2 r_2 + \alpha(r_4 - r_1) - r_3) = k_2 - k_1.$$

(Notice that zero is a singular value for the normalization.) Recall that $k_2 - k_1 = \pm(\Delta)^{1/2}$, depending on the choice. We can exchange k_1 and k_2 if locally this equation cannot be solved. (In fact, we can also choose any multiple of $k_2 - k_1$ instead of $k_2 - k_1$ and obtain similar results.) Thus, with a proper choice, this equation can be generically solved.

Complex case.

Assume now that we are in the complex case. In order to find β we can choose several different normalizations. The one we will choose here is to make the (1,1) entry of the equation equal to the (2,2) entry. The corresponding equation is

$$\alpha\beta r_2 + \frac{1}{2}(\alpha + \beta)(r_4 - r_1) + r_3 = 0.$$

After substituting the value of α and simplifying, we obtain a quadratic equation for β , which can be locally solved in generic cases. As before, $k_1 = \frac{1}{2}(s_1 + s_4)$ and $k_2 = \pm\frac{1}{2}(-\Delta)^{1/2}$, depending on the choice. Different normalizations will be needed for those generic cases for which the equation cannot be solved. Later on we will need to change this frame into a more convenient one.

This completes the calculation of the moving frame and we can now find its associated Maurer-Cartan matrix. Unlike the classical case, the entries of the Maurer-Cartan matrix will produce a complete set of independent and generating differential invariants (see [7]). But before, we will write a final description of the generating invariants that can be obtained directly from this normalization.

Theorem 1. *A generic curve in $\text{Gr}_2(\mathbb{R}^4)$ has a system of four functionally independent and generating differential invariants, two of order three and one of each order 4 and 5.*

Proof. Standard arguments that can be found in [5] tell us that, in the case of real Schwarzian eigenvalues, the diagonal of i_4 in (3.13) equals the derivative of the diagonalization of $S(u)$. In the complex case, the same arguments show that, if

$$i_3 = \begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}$$

then i_4 in (3.13) will be of the form

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} (k_1)_x & -(k_2)_x \\ (k_2)_x & (k_1)_x \end{pmatrix}.$$

In the real case, (3.13) adds one independent fourth order invariant to our group of third order ones, namely the $(2, 1)$ entry of (3.13) (which has not been normalized). In the complex case, our normalization implies $\alpha_1 = 0$. Hence, β_1 will be the additional functionally independent fourth order invariant.

One more fifth order invariant exists, one that is not a function of the previous three invariants and their derivatives. The description of this generator can be found following the normalization process in [5]. If we write the fifth order normalization equations as above, there is one entry that corresponds to a previously normalized entry in i_4 , namely $(1, 2)$ in the real case and $(2, 1)$ in the complex one. That entry is an independent fifth order differential invariant. \square

The explicit expression is not relevant now. Indeed we will make use of a different, but more convenient, choice.

3.2. Grassmannian Maurer-Cartan matrix associated to the left moving frame.

Definition 2. Let ρ be a left moving frame. The matrix

$$K = \rho^{-1} \rho_x$$

is called the (left) Maurer-Cartan matrix associated to ρ (it is the horizontal component of the pull-back by ρ of the Maurer-Cartan form of the group G). The entries of K are clearly differential invariants; a Theorem in [7] states that they generate all other differential invariants for the curve. Since the right moving frame associated to a left one is its inverse, the right Maurer-Cartan matrix is the negative of the left one.

Some of the work in [5] provides what are normally called horizontal recurrence formulas. These formulas can be used to recurrently calculate the entries of K . The following Proposition is a reformulation of these recurrence formulas for the case at hand.

Proposition 2. *Let K be the left Maurer-Cartan matrix associated to a moving frame ρ . Assume that $\rho \cdot u_s = i_s$, $s = 0, 1, 2, \dots$. Then*

$$(3.15) \quad K \cdot i_s = i_{s+1} - (i_s)_x$$

$s = 0, 1, 2, \dots$, where the dot in $K \cdot i_s$ represents the s^{th} infinitesimal prolonged action of the Lie algebra on the element of the jet space i_s .

We will use this recurrence relation to find the left Maurer-Cartan matrix associated to the left moving frame $\rho = g^{-1}$, where g was found in our previous section. Denote the Maurer-Cartan matrix by

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

If

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \in \mathfrak{g}$$

we can calculate the infinitesimal version of (3.1) to be

$$V \cdot u = V_{11}u - uV_{22} - uV_{21}u + V_{12}.$$

Given that $i_0 = 0$ and $i_1 = I$ (see (3.2)), for $s = 0$ (3.15) becomes

$$K \cdot 0 = K_{12} = I.$$

The infinitesimal version of the action in (3.2) is given by

$$V \cdot u^{(1)} = V_{11}u_1 - u_1V_{22} - u_1V_{21}u - uV_{21}u_1.$$

Now, $i_2 = 0$, so for $s = 1$ formula (3.15) becomes

$$K \cdot i_1 = K_{11} - K_{22} = i_2 - (i_1)_x = 0, \quad K_{11} = K_{22}$$

Again, the infinitesimal version of the action in (3.3) is given by

$$V \cdot u^{(2)} = V_{11}u_2 - u_2V_{22} - 2u_1V_{21}u_1 + R_2$$

where R_2 are terms that vanish whenever $u = 0$. From here, when $s = 2$, formula (3.15) becomes

$$K \cdot i_2 = -2K_{21} = i_3, \quad K_{21} = -\frac{1}{2}i_3.$$

Finally, the infinitesimal version of the action in (3.4) is given by

$$V \cdot u^{(3)} = V_{11}u_3 - u_3V_{22} - 3(u_2V_{21}u_1 + u_1V_{21}u_2) + R_3$$

where R_3 vanishes whenever $u = 0$. Therefore, for $s = 3$, equation (3.15) becomes

$$(3.16) \quad K_{11}i_3 - i_3K_{11} = i_4 - (i_3)_x.$$

Real case.

In the case of real Schwarzian eigenvalues, both sides of (3.16) have a vanishing diagonal, and so, the off-diagonal entries of K_{11} are determined by the entries of i_4 . That is, if

$$K_{21} = -\frac{1}{2}i_3 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

then, straightforward calculations show that the off-diagonal entries of K_{11} are given by

$$\begin{pmatrix} 0 & 1 \\ \kappa_3 & 0 \end{pmatrix}$$

where κ_3 is an independent fourth order differential invariant determined by the one appearing in i_4 through (3.16). One can find explicitly κ_3 , but we do not really need its expression here and we will change frames soon. Notice that the choice of $k_2 - k_1$ in the normalization of (3.14) implies that the $(1, 2)$ entry of K_{11} is constant and equal to 1. Recall that zero was a singular value.

We do not need to go to the fourth order. Indeed, since $K_{11} = K_{22}$, we do know that the traces of both matrices vanish. Therefore, only one unknown entry of K remains, namely the one entry determining its trace. On the other hand, we do know that the entries of K generate all differential invariants, and there is one fifth order invariant that is unaccounted for. Therefore, we can conclude that the missing entry, κ_4 , is such a generator without actually calculating it explicitly. If we were to need its explicit formula, we could use the recurrence formulas further to find them. Thus, in the real case

$$K_{11} = \begin{pmatrix} \kappa_4 & 1 \\ \kappa_3 & -\kappa_4 \end{pmatrix}.$$

Complex case.

In the case of complex Schwarzian eigenvalues

$$i_3 = \begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}, \quad K_{21} = -\frac{1}{2}i_3 = \begin{pmatrix} \kappa_1 & -\kappa_2 \\ \kappa_2 & \kappa_1 \end{pmatrix}.$$

As we pointed out before, i_4 can be split into

$$i_4 = \begin{pmatrix} \alpha_1 + (k_1)_x & -(k_2)_x \\ (k_2)_x & -\alpha_1 + (k_1)_x \end{pmatrix}$$

since, with our normalization, $\beta_1 = 0$. If $K_{11} = (k_{ij})$, then (3.16) becomes

$$\begin{pmatrix} k_2(k_{12} + k_{21}) & -2k_{11}k_2 \\ -2k_2k_{11} & -k_2(k_{12} + k_{21}) \end{pmatrix} = \begin{pmatrix} 0 & \beta_1 \\ \beta_1 & 0 \end{pmatrix}$$

and so

$$K_{11} = \begin{pmatrix} \kappa_3 & -c \\ c & -\kappa_3 \end{pmatrix}$$

where c is still to be found and κ_3 is a fourth order invariant obtained from α_1 . As before, c needs to be the missing fifth order generator κ_4 , since it is the only undetermined entry. Finally

$$K_{11} = \begin{pmatrix} \kappa_3 & -\kappa_4 \\ \kappa_4 & -\kappa_3 \end{pmatrix}.$$

We have finally proved the following theorem.

Theorem 2. *Let ρ be the left moving frame determined in the previous section. Its Maurer-Cartan matrix is given by*

$$(3.17) \quad K = \rho^{-1}\rho_x = \begin{pmatrix} \kappa_4 & 1 & 1 & 0 \\ \kappa_3 & -\kappa_4 & 0 & 1 \\ \kappa_1 & 0 & \kappa_4 & 1 \\ 0 & \kappa_2 & \kappa_3 & -\kappa_4 \end{pmatrix}$$

in the generic case of real eigenvalues, and by

$$(3.18) \quad K = \begin{pmatrix} \kappa_3 & -\kappa_4 & 1 & 0 \\ \kappa_4 & -\kappa_3 & 0 & 1 \\ \kappa_1 & -\kappa_2 & \kappa_3 & -\kappa_4 \\ \kappa_2 & \kappa_1 & \kappa_4 & -\kappa_3 \end{pmatrix}$$

in the generic case of complex eigenvalues. In both cases κ_1 and κ_2 are third order differential invariants, κ_3 is fourth order and κ_4 is fifth order. The invariants κ_i , $i = 1, \dots, 4$ form a generating system of independent differential invariants for generic curves of Grassmannians.

3.3. Invariant evolutions of Grassmannian curves. Once a group-based moving frame has been found, a theorem in [12] provides us with a classical moving frame (an invariant curve in the frame bundle over the curve). This classical frame can be used to write the most general form of an invariant evolution of curves in M , i.e., an evolution for which the group takes solutions to solutions. The following theorem explains what that evolution is in our case. The theorem is a direct consequence of the results in [12].

Theorem 3. *Let $\rho = g^{-1}$ be the left moving frame obtained in section 3.1, and let B be the matrix defining g . Then, the most general form of an evolution of Grassmannian curves, invariant under the action (3.1), is given by*

$$(3.19) \quad u_t = u_1 B^{-1} \mathbf{r} B$$

where \mathbf{r} is any 2×2 matrix depending on κ_i , $i = 1, \dots, 4$, and their derivatives.

This expression is the particular form in our example of the more general formula for $|1|$ -graded Lie algebras $\rho_{-1}^{-1}(\rho_{-1})_t = \text{Ad}(\rho_0)(\mathbf{r})$, where $\mathbf{r} \in \mathfrak{g}_{-1}$ is an element of \mathfrak{g}_{-1} depending on differential invariants of the curve, and where $\rho = \rho_{-1}\rho_0\rho_1$ is a left moving frame factored out following the gradation, with ρ_{-1} defining our section (for more details see [12]). In our case

$$\rho_{-1} = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix}, \quad \rho_0 = \begin{pmatrix} u_1 B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} I & 0 \\ -Z & I \end{pmatrix}.$$

The matrix \mathbf{r} has a Hamiltonian interpretation that will be explained in our next section. Because of the Hamiltonian interpretation, the following Theorems are of interest to us. They describe the behavior of evolutions as non-Schwarzian invariants κ_3, κ_4 vanish, with \mathbf{r} having the same normal form as $S(u)$. Since $BS(u)B^{-1} = i_3 = -2K_{12}$, the particular choice $\mathbf{r} = K_{12}$ corresponds to the flow

$$u_t = u_1 B^{-1} K_{12} B = -\frac{1}{2} u_1 S(u)$$

which is a generalization of the well-known *KdV-Schwarzian evolution* to Grassmannian flows.

Theorem 4. *Assume $S(u)$ has real eigenvalues and assume \mathbf{r} is a diagonal matrix. Then, the level set $\kappa_3 = 0$ is invariant under evolution (3.19).*

For general diagonal \mathbf{r} , and in particular when $\mathbf{r} = K_{12}$, the level set $\kappa_4 = 0$ is not invariant under evolution (3.19).

Assume $S(u)$ has complex eigenvalues, and assume $\mathbf{r} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is in normal form. Then, for general \mathbf{r} (and in particular if $\mathbf{r} = K_{12}$), the flow blows up as $\kappa_3 \rightarrow 0$. The level set $\kappa_4 = \kappa_3 = 0$ is, therefore, not preserved.

The relevance of the second part of this proposition will be better understood in our next section. The situation described in this theorem is very similar to the one for spinor curves ([14]) for which it was proved that *any* choice of local moving frame would induce this situation.

Proof. Let $u(t, x)$ be a flow solution of (3.19) with \mathbf{r} diagonal. A theorem in [12] (or a straightforward calculation) shows that $N = \rho^{-1}\rho_t$ is of the form

$$N = \begin{pmatrix} N_{11} & \mathbf{r} \\ N_{21} & N_{22} \end{pmatrix}.$$

Furthermore, since $\frac{d}{dx}$ and $\frac{d}{dt}$ commute, compatibility of $\rho_x = \rho K$ and $\rho_t = \rho N$ imply

$$K_t = N_x + [K, N].$$

This breaks up into a number of individual equations (according to the gradation of the algebra), namely

$$(3.20) \quad 0 = \mathbf{r}_x + N_{22} - N_{11} + [K_{11}, \mathbf{r}]$$

$$(3.21) \quad 0 = (N_{11} - N_{22})_x + [K_{11}, N_{11} - N_{22}] - \mathbf{r}K_{21} - K_{21}\mathbf{r} + 2N_{21}$$

$$(3.22) \quad (K_{11})_t = (N_{11})_x + [K_{11}, N_{11}] - \mathbf{r}K_{21} + N_{21}$$

$$(3.23) \quad (K_{21})_t = (N_{21})_x + K_{21}N_{11} - N_{22}K_{21} + [K_{11}, N_{21}].$$

Real case.

Assume now that $S(u)$ has real eigenvalues and that $\mathbf{r} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ is diagonal.

Let us call

$$N_{11} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad N_{22} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then, using (3.20) we obtain

$$(3.24) \quad N_{11} - N_{22} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{pmatrix} = \begin{pmatrix} (r_1)_x & r_2 - r_1 \\ \kappa_3(r_1 - r_2) & (r_2)_x \end{pmatrix}.$$

Using (3.21) we have

$$(3.25) \quad N_{21} = \begin{pmatrix} -\frac{1}{2}r_1'' + \kappa_1 r_1 & (r_1 - r_2)' + \kappa_4(r_1 - r_2) \\ (\frac{1}{2}(\kappa_3)' - \kappa_4\kappa_3)(r_1 - r_2) + \kappa_3(r_1 - r_2)' & -\frac{1}{2}r_2'' + r_2\kappa_2 \end{pmatrix}.$$

From here, (3.23) becomes

$$(3.26) \quad \begin{pmatrix} (\kappa_1)_t & 0 \\ 0 & (\kappa_2)_t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}r_1''' + (r_1\kappa_1)' + \kappa_1 r_1' & \kappa_1 b_1 - \kappa_2 b_2 + \frac{3}{2}(r_1 - r_2)'' + \kappa_2 r_2 - \kappa_1 r_1 \\ \kappa_2 c_1 - \kappa_1 c_2 & -\frac{1}{2}r_2''' + (r_2\kappa_2)' + \kappa_2 r_2' \end{pmatrix} + X_1$$

where X_1 is a matrix whose explicit expression can be directly found, and whose relevant property is that it is well-defined for κ_3 and κ_4 sufficiently small, X_1 becomes strictly upper triangular as $\kappa_3 \rightarrow 0$, and $X_1 \rightarrow 0$ as $\kappa_3, \kappa_4 \rightarrow 0$.

Therefore, (3.26) determines completely the evolution of κ_1 and κ_2 . Notice that the choice $r_i = \kappa_i$ produces a decoupled system of KdV equations in κ_1 and κ_2 in the limit. From (3.24) we have that, as $\kappa_3 \rightarrow 0$, $c_1 = c_2$, which together with (3.26) gives us, as $\kappa_3 \rightarrow 0$, (and as long as $\kappa_1 - \kappa_2 \neq 0$)

$$c_1 = c_2 = 0.$$

Equations (3.24) and (3.26) also provide the explicit expression for b_1 and b_2 , these are

$$(3.27) \quad b_1 = -\frac{3}{2} \frac{(r_1 - r_2)''}{\kappa_1 - \kappa_2} + \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} r_1 - 2 \frac{\kappa_2}{\kappa_1 - \kappa_2} r_2 + Y_1$$

$$(3.28) \quad b_2 = -\frac{3}{2} \frac{(r_1 - r_2)''}{\kappa_1 - \kappa_2} + 2 \frac{\kappa_1}{\kappa_1 - \kappa_2} r_1 - \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} r_2 + Y_2$$

where $Y_i \rightarrow 0$ as $\kappa_4 \rightarrow 0$. With these values, equation (3.22) proves the proposition. The entry (2, 1) of (3.22) shows $(\kappa_3)_t = 0$ as $\kappa_3 \rightarrow 0$ (independently of κ_4), and hence $\kappa_3 = 0$ is preserved by the evolution. Finally, the (1, 2) entry of (3.22) will determine the value of $d_1 - a_1$, and we know that $a_1 - a_2 = (r_1)_x$, $d_1 - d_2 = (r_2)_x$ and $a_1 + a_2 + d_1 + d_2 = 0$ (from the algebra condition). These relations completely

determine all values of N . If we further assume that $\kappa_4 \rightarrow 0$, after straightforward calculations the remaining values show that

$$(\kappa_4)_t = \frac{1}{2}b_1'' + \frac{1}{4}(r_1'' - r_2'')$$

and so $\kappa_4 = 0$ is not, in general, preserved, even in the case $r_i = \kappa_i$.

Complex case.

Assume now that $S(u)$ has complex eigenvalues, and assume that

$$\mathbf{r} = \begin{pmatrix} r_1 & -r_2 \\ r_2 & r_1 \end{pmatrix}.$$

Using the same equations as in the real case, we obtained the values

$$N_{11} - N_{22} = \begin{pmatrix} r_1' + r_2 & -r_2' - 2\kappa_3 r_2 \\ r_2' - 2\kappa_3 r_2 & r_1' - r_2 \end{pmatrix}$$

$$N_{21} = \begin{pmatrix} -\frac{1}{2}r_1'' - r_2' + r_1\kappa_1 - r_2\kappa_2 & \frac{1}{2}r_2'' - (r_1\kappa_2 + r_2\kappa_1) \\ -\frac{1}{2}r_2'' + r_2\kappa_1 + r_1\kappa_2 & -\frac{1}{2}r_1'' + r_2' + r_1\kappa_1 - r_2\kappa_2 \end{pmatrix} + Z_1$$

where

$$Z_1 = \begin{pmatrix} -2\kappa_3\kappa_4 r_2 + \kappa_3 r_2 & -r_2\kappa_4 + \kappa_3(r_2' + 2\kappa_3 r_2) + (r_2\kappa_3)' \\ -r_2\kappa_4 + \kappa_3(r_2' - 2\kappa_3 r_2) + (\kappa_3 r_2)' & 2\kappa_3\kappa_4 r_2 - \kappa_3 r_2 \end{pmatrix}.$$

The t -evolution of K_{12} in (3.20) imposes some conditions on the entries of LHS, namely the (1, 1) and (2, 2) entries must be equal, and the (1, 2) and (2, 1) entries must be the negative of each other. This leads to solving for $c_1 + b_1$ and for $a_1 - d_1$, which are determined by

$$\kappa_2(c_1 + b_1) = z_1, \quad \kappa_2(a_1 - d_1) = z_2$$

where z_1 and z_2 are well defined as κ_3 and κ_4 vanish, and they both vanish in the limit. Recall that the trace of N is zero. Therefore $a_1 + d_1 + a + 2 + d_2 = 2(a_1 + d_1) - 2r_1' = 0$. From here $a_1 + d_1 = r_1'$ and, in the limit, $a_1 = \frac{1}{2}r_1'$. From here we obtain the evolutions for κ_1 and κ_2 , namely

$$(3.29) \quad \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}D^3 + D\kappa_1 + \kappa_1 D & -D\kappa_2 - \kappa_2 D \\ D\kappa_2 + \kappa_2 D & -\frac{1}{2}D^3 + D\kappa_1 + \kappa_1 D \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where, again, y_1 and y_2 are well defined and vanish as κ_3, κ_4 vanish. In the limit, the evolution becomes a complexly coupled system of KdV equations, evolution that also appears in the conformal case. Still, such a limit creates a singularity in the flow. A final condition is imposed by the evolution of K_{11} in (3.22), where the sum of the (1, 2) and (2, 1) entries vanish. The condition gives us an equation that allows us to finally solve for c_1 and b_1 (we can solve for all the other ones with the data we have up to this point). The equation is

$$(c_1 + b_1)' + 2\kappa_3(b_1 - c_1) - 2(d_1 - a_1)\kappa_4 - 2\kappa_4 r_2 + 2\kappa_3 r_2' + 2(\kappa_3 r_2)' = 0.$$

The expressions for $d_1 - a_1$ and $c_1 + b_1$ tells us that b_1 and c_1 are not well-defined in the limit. Therefore, the t -evolution of κ_4 also blows up as $\kappa_3 \rightarrow 0$. Finally, one can check that the level set $\kappa_3 = 0$ is preserved and $(\kappa_3)_t = 0$ as $\kappa_4 \rightarrow 0$. We leave the rest of the details to the reader. \square

4. GEOMETRIC HAMILTONIAN STRUCTURES

There are two well-known Poisson structures defined on the space of Loops on the dual of a semisimple Lie algebra. If $\mathcal{H}, \mathcal{F} : \mathcal{L}\mathfrak{g}^* \rightarrow \mathbb{R}$ are two operators, their Poisson brackets are given by new operators on $\mathcal{L}\mathfrak{g}^*$ defined as

$$(4.1) \quad \{\mathcal{H}, \mathcal{F}\}(L) = \int_{S^1} \langle B \left(\frac{\delta \mathcal{H}}{\delta L} \right)_x + \text{ad}^* \left(\frac{\delta \mathcal{H}}{\delta L} \right)(L), \frac{\delta \mathcal{F}}{\delta L} \rangle dx$$

$$(4.2) \quad \{\mathcal{H}, \mathcal{F}\}_0(L) = \int_{S^1} \langle \text{ad}^* \left(\frac{\delta \mathcal{H}}{\delta L} \right) (L_0), \frac{\delta \mathcal{F}}{\delta L} \rangle dx$$

where B is any bilinear identification of \mathfrak{g} with its dual and where $L_0 \in \mathfrak{g}$ is any constant element. The elements $\frac{\delta \mathcal{H}}{\delta L}, \frac{\delta \mathcal{F}}{\delta L} \in \mathfrak{g}$ are the variational derivatives at L and \langle, \rangle is the invariant pairing of \mathfrak{g} with \mathfrak{g}^* . The usual choice is to identify \mathfrak{g} with its dual \mathfrak{g}^* using the Killing form, or, for example the trace of the product if $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$. With these standard choices, if $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$, then the dual to the entry (i, j) is given by the entry (j, i) (or a multiple of it).

For a given choice of normalization equations, let's denote the space of Maurer-Cartan matrices by \mathcal{K} . The author of [12] showed that, locally around a generic curve $u(x) \in G/H$ with G semisimple, the space \mathcal{K} can be written as a quotient $U/\mathcal{L}H$ where $U \subset \mathcal{L}\mathfrak{g}^*$ is open. She also showed that the Poisson bracket (4.1) could then be reduced to the quotient to produce what she called a Geometric Poisson structure, a Hamiltonian structure on the space of differential invariant of curves. The Poisson bracket (4.2) cannot always be reduced, but its reduction usually indicates the existence of a curve evolution inducing a completely integrable system on its invariants. In this case, the integrable system is biHamiltonian with respect to both reduced brackets. The paper [12] also showed how given *any* evolution Hamiltonian with respect to the geometric Poisson bracket, one could always find a geometric realization as invariant flows in G/H . The coefficients of the realization were explicitly related to the Hamiltonian functional.

Both the reduction of (4.1) and the geometric realizations of Hamiltonian evolutions can be found explicitly in most cases, under some minimal conditions. For example, the reduction process can be described in the following terms: given $h, f : \mathcal{K} \rightarrow \mathbb{R}$, we extend them to functionals $\mathcal{H}, \mathcal{F} : \mathcal{L}\mathfrak{g}^* \rightarrow \mathbb{R}$ so that the extensions are constant on the $\mathcal{L}H$ -leaves along \mathcal{K} . This condition is infinitesimally described by stating

$$(4.3) \quad \left(\frac{\delta \mathcal{H}}{\delta L} \right)_x + \text{ad}(K) \left(\frac{\delta \mathcal{H}}{\delta L} \right) \in \mathfrak{h}^0,$$

with $K \in \mathcal{K}$. If two such extensions can be found, then the reduced Poisson bracket is given by $\{h, f\}_R(\mathbf{k}) = \{\mathcal{H}, \mathcal{F}\}(K)$, where K is the Maurer-Cartan matrix defining the generating invariants \mathbf{k} . The infinitesimal condition usually allows us to solve explicitly for $\frac{\delta \mathcal{H}}{\delta L}$.

Perhaps the most interesting part is the direct relation between invariant evolutions and evolutions that are Hamiltonian with respect to the reduction of (4.1). The following theorem can be found in [12].

Theorem 5. *Assume an evolution*

$$(4.4) \quad \mathbf{k}_t = F(\mathbf{k}, \mathbf{k}_x, \mathbf{k}_{xx}, \dots)$$

is Hamiltonian with respect to the geometric Poisson bracket obtained when reducing (4.1) to \mathcal{K} . Assume f is its Hamiltonian functional and \mathcal{F} is an extension constant on the LH -leaves along \mathcal{K} . Assume $\rho = \rho_{-1}\rho_0\rho_1$ according to a $|1|$ -grading of the algebra and assume we can identify ρ_{-1} with a nondegenerate curve u in G/H . Let $\mathbf{r} = \left(\frac{\delta\mathcal{F}}{\delta L}(K)\right)_{-1}$. Then

$$(4.5) \quad \rho_{-1}^{-1}(\rho_{-1})_t = Ad(\rho_0)\mathbf{r}$$

is a geometric realization for (4.4).

This direct relation, together with Theorem 4, tells us that the reduction of (4.1) cannot be further restricted to the level set $\kappa_3 = \kappa_4 = 0$. On the other hand, such a reduction was possible for the conformal sphere of signature $(n, 0)$, and, when reduced, one would obtain BiHamiltonian structures for a complexly coupled system of KdV equations on κ_1 and κ_2 (see [14]). The level set $\kappa_3 = \kappa_4 = 0$ was also preserved by invariant evolutions whenever \mathbf{r} was in the same normal form as $S(u)$. To achieve this restriction in the conformal case, one needed to use *natural conformal moving frames*. In our next section we study the local geometry of conformal curves on the sphere of signature $(2, 2)$, followed by the definition of natural frames. Natural moving frames are easier to understand geometrically in the conformal picture than in the Grassmannian one, although they correspond algebraically, so our strategy is to shift the geometric knowledge we have in the conformal case to the Grassmannian one.

5. THE LOCAL GEOMETRY OF CONFORMAL CURVES

In this section we will describe the analogous information presented in the previous section for the case of the conformal Möbius sphere of signature $(2, 2)$. Although the case of the conformal Möbius sphere of signature $(n, 0)$ was studied in [13], the change of signature and the need to relate it to the Grassmannian case urge us to use normalization equations, a different approach from that in the original paper. The normalization constants will be matched to the ones used for the Grassmannian case, while trying to reproduce the results in [14] for the general conformal case. Although the isomorphism Φ in section 2 guarantees our process, we need to have explicit descriptions to relate it to the natural frame.

Following the gradation of $\mathfrak{so}(3, 3)$ we described in our second section, an element $g \in \text{SO}(3, 3)$ can be locally factored as

$$g = \begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ -\frac{1}{2}\|z\|^2 & -z^t J & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \Theta & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & -y^t J & -\frac{1}{2}\|y\|^2 \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}$, $\Theta \in O(2, 2)$, $O(2, 2)$ is represented as the group that preserves

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and where $\|v\|^2 = v^t J v$ (the notation is deceiving since it can be negative). We will also denote $\langle v, w \rangle = v^t J w$. If we identify $\text{SO}(3, 3)/P_C$ with the section $\alpha = 1, z =$

$0, \Theta = I$, that is, with

$$g_u = \begin{pmatrix} 1 & -u^t J & -\frac{1}{2}\|u\|^2 \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix}$$

then $g \cdot u$ is completely determined by the relation

$$gg_u = g_{g \cdot u} h$$

with $h \in P_C$. The relation uniquely determines

$$(5.1) \quad g \cdot u = \frac{\Theta(u+y) - \frac{\alpha}{2}\|u+y\|^2 z}{\alpha^{-1} - z^t J \Theta(u+y) + \frac{\alpha}{4}\|z\|^2 \|u+y\|^2}.$$

5.1. Group-based moving frame for conformal generic curves. Let us denote denominator and numerator by $g \cdot u = \frac{F}{M}$. The normalization process below uses normalization constants that are coordinated with those of the Grassmannian. Consider the vectors $e_1 = (1, 0, 0, 1)^t$, $e_2 = (0, 1, 1, 0)^t$, $e_3 = (0, 1, -1, 0)^t$, $e_4 = (1, 0, 0, -1)^t$ so that $\|e_1\|^2 = \|e_2\|^2 = -\|e_3\|^2 = -\|e_4\|^2 = 2$.

Zeroth normalization equation. This equation is given by

$$g \cdot u = i_0 = 0$$

which can be readily solved with the choice $u+y=0$ or $y=-u$.

First normalization equation. Since $g \cdot u_1 = \frac{F_1}{M} - \frac{F}{M} \frac{M_1}{M}$. when substituting $g \cdot u = \frac{F}{M} = 0$ we get that the first normalization equation is

$$g \cdot u_1 = \frac{F_1}{M} = \alpha \Theta u_1 = i_1 = e_3.$$

This determines $\alpha^2 \|u_1\|^2 = -2$ and imposes conditions on Θ . If $\|u_1\|^2 > 0$ (notice the abuse of notation since $\|\cdot\|^2$ can be negative), then this choice is not possible and both Grassmannian and conformal cases will need to be renormalized to adjust to the situation (by, for example, choosing $e_2 = i_1$ and $i_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in the Grassmannian case). There is no problem doing so and we obtain an analogous result.

Second normalization equation. Since

$$g \cdot u_2 = \frac{F_2}{M} - 2 \frac{F_1}{M} \frac{M_1}{M} - \frac{F}{M} \left(\frac{M_2}{M} - 2 \frac{M_1^2}{M^2} \right)$$

after substituting previous normalizations we obtain

$$g \cdot u_2 = \alpha \Theta u_2 + 2z + 2(z^t J e_3) e_3 = i_2 = 0.$$

This choice of i_2 allows us to solve for z in terms of Θ (if $z = (z_1, z_2, z_3, z_4)^t$, then $(z_1, z_3, z_2, z_4)^t = -\frac{1}{2} \alpha \Theta u_2$). With this value for z , one can check directly that $\|z\|^2 = \frac{1}{4} \|u_2\|^2 - \frac{\alpha^2}{4} \langle u_1, u_2 \rangle^2$ and $z^t J e_3 = \frac{\alpha}{2} \langle u_1, u_2 \rangle$.

Third normalization equation. Given that

$$g \cdot u_3 = \frac{F_3}{M} - 3 \frac{M_1}{M} \left(\frac{F_2}{M} - 2 \frac{F_1 M_1}{M^2} \right) - 3 \frac{M_2}{M} \frac{F_1}{M} - \frac{F}{M} \left(\frac{M_3}{M} - 6 \frac{M_2 M_1}{M^2} + 6 \frac{M_1^3}{M^3} \right)$$

substituting previous normalizations yields the equation

$$\alpha \Theta u_3 - 3\alpha^2 \langle u_1, u_2 \rangle z - 6(z^t J e_3)^2 e_3 - 3\|z\|^2 e_3 = i_3.$$

If we write z and e_3 in terms of Θ using previous normalization equations, we arrive to the third normalization equation

$$(5.2) \quad \alpha\Theta \left(u_3 - 3 \frac{\langle u_1, u_2 \rangle}{\|u_1\|^2} u_2 + \frac{3 \|u_2\|^2}{2 \|u_1\|^2} u_1 \right) = \alpha\Theta G_3 = i_3$$

where the vector G_3 is defined uniquely by the equation.

As expected, this equation has rank 2. In fact,

$$(5.3) \quad \langle i_3, i_1 \rangle = -2 \left(\frac{\langle u_1, u_3 \rangle}{\|u_1\|^2} - 3 \frac{\langle u_1, u_2 \rangle^2}{\|u_1\|^4} + \frac{3 \|u_2\|^2}{2 \|u_1\|^2} \right) = -2I_1$$

(we define I_1 through this equation) and $\|i_3\|^2 = -2(I_2 + I_1^2)$ where

$$(5.4) \quad I_2 = \frac{\langle u_3, u_3 \rangle}{\|u_1\|^2} - 6 \frac{\langle u_1, u_2 \rangle \langle u_2, u_3 \rangle}{\|u_1\|^4} - \frac{\langle u_1, u_3 \rangle^2}{\|u_1\|^4} + 6 \frac{\langle u_1, u_2 \rangle^2 \langle u_1, u_3 \rangle}{\|u_1\|^6} + 9 \frac{\langle u_1, u_2 \rangle^2 \langle u_2, u_2 \rangle}{\|u_1\|^6} - 9 \frac{\langle u_1, u_2 \rangle^4}{\|u_1\|^8}.$$

Both I_1 and I_2 appeared in [13] for the case of signature $(n+1, 1)$ and they were called in [14] *differential invariants of Schwarzian or projective type*. If ϕ is a change of parameter, one can check that

$$\phi^* I_1 = (\phi')^2 I_1 \circ \phi + S(\phi), \quad \phi^* I_2 = (\phi')^4 I_2 \circ \phi$$

where $S(\phi)$ denotes the Schwarzian derivative of ϕ . The invariant $I_2^{1/4}$ is used to define the conformal arc-length.

Now we need to make a choice for i_3 . The value of i_3 in the Grassmannian case depends on whether the Schwarzian derivative of u has real or complex eigenvalues. Here we will make two different choices to match the Grassmannian choices. We will also call them the real and complex cases, although this relation is only apparent through its connection to the Grassmannian.

Real case. In this case we will choose

$$(5.5) \quad i_3 = \begin{pmatrix} 0 \\ * \\ * \\ 0 \end{pmatrix}.$$

That is, we will ask $\alpha\Theta G_3$ to lie on the plane generated by e_2 and e_3 . Assume $i_3 = \hat{k}_1 e_2 + \hat{k}_2 e_3$. Given that $\alpha\Theta u_1 = e_3$, this will determine the value of $\Theta^{-1} e_3$ and that of $\Theta^{-1} e_2$ (with $\Theta \in O(2, 2)$ being guarantee by the entries that are not normalized). We need to find one more piece of information about $\Theta^{-1} e_1$ or e_4 to completely determine Θ . We can accomplish that in the fourth normalization equation, where only one more equation needs to be normalized.

Complex case. In the complex case we will choose

$$i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_4$$

so that we will have determined $\Theta^{-1} e_3$ and $\Theta^{-1} e_4$. Again, one more normalization in the fourth order equation will completely determine Θ .

Fourth normalization equation. As done with the third equation, we will need to differentiate once more and substitute the previous normalizations. Details are dull and do not add information, so we will leave them up to the interested reader. After substituting the values for z and e_3 in terms of $\alpha\Theta$, once more the fourth normalization equation will look like

$$\alpha\Theta G_4 = i_4$$

where G_4 is a vector depending exclusively on derivatives of u .

Real case. In this case we will ask i_4 to belong to the subspace generated by e_2 , e_3 and e_4 ; this forces the coefficient of e_1 to vanish, which is the last normalization equation we need to use to complete the determination of the moving frame ρ . Notice that these choices can be accomplished generically (to be sure, we would need to find G_4 , which is straightforward but long and space consuming so we will not include it in the paper).

Complex case. In this case we will ask for the same condition on i_4 , which will also imply that the e_1 coefficient of ΘG_4 will vanish. This normalization determines $\Theta^{-1}e_2$ additionally to $\Theta^{-1}e_3$ and $\Theta^{-1}e_4$, so the frame is now completely determined.

5.2. Grassmannian Maurer-Cartan matrix associated to the moving frame.

In this section we will use Proposition 2 to find the explicit form of the Maurer-Cartan matrix K associated to the moving frame ρ that we just found. Let $v \in \mathfrak{so}(3,3)$ and assume we split it as

$$(5.6) \quad v = \begin{pmatrix} v_\alpha & -v_{-1}^t J & 0 \\ v_1 & v_0 & v_{-1} \\ 0 & -v_1^t J & -v_\alpha \end{pmatrix}.$$

Then, the prolonged zero, first, second and third order infinitesimal action of the algebra on the manifold are given by

$$\begin{aligned} v \cdot u &= v_0 \cdot u - \frac{1}{2} v_1 \|u\|^2 + v_{-1}, \\ v \cdot u_1 &= v_0 u_1 + v_\alpha u_1 + R_1, \\ v \cdot u_2 &= v_\alpha u_2 + v_0 u_2 - \|u_1\|^2 v_1 + 2u_1 v_1^t J u_1 + R_2, \\ v \cdot u_3 &= v_\alpha u_3 + v_0 u_3 - 3\langle u_1, u_2 \rangle v_1 + 3(v_1^t J u_1) u_2 + 3(v_1^t J u_2) u_1 + R_3, \end{aligned}$$

where R_i all contain terms that vanish as u vanishes. Let K be the Maurer-Cartan matrix in the conformal case and assume we split K as done in (5.6). Using these equations for the infinitesimal action in Proposition 2 we get the following information on K

$$\begin{aligned} K \cdot i_0 &= K_{-1} = i_1 = e_3, \\ K \cdot i_1 &= K_0 i_1 + K_\alpha i_1 = (K_0 + K_\alpha) e_3 = 0, \\ K \cdot i_2 &= 2K_1 - 2(K_1^t J e_3) e_3 = i_3, \\ K \cdot i_3 &= K_0 i_3 + K_\alpha i_3 = i_4 - (i_3)_x. \end{aligned}$$

The first equation determines K_{-1} and the second tells us that $K_\alpha = 0$ and

$$(5.7) \quad K_0 = \begin{pmatrix} a & b & b & 0 \\ c & 0 & 0 & -b \\ c & 0 & 0 & -b \\ 0 & -c & -c & -a \end{pmatrix}$$

where the entries still need to be determined. The last equation will determine the value of K_1 , depending on whether we are in a real or a complex case, as expected.

In the real case, if $i_3 = \hat{k}_1 e_2 + \hat{k}_2 e_3$, then

$$K_1 = \begin{pmatrix} 0 \\ k_1 \\ k_2 \\ 0 \end{pmatrix}$$

where $k_1 = \frac{1}{2}(\hat{k}_1 - \hat{k}_2)$ and $k_2 = \frac{1}{2}(\hat{k}_1 + \hat{k}_2)$.

In the complex case, if $i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_4$, then

$$K_1 = \begin{pmatrix} -k_2 \\ k_1 \\ -k_1 \\ k_2 \end{pmatrix}$$

where $k_i = -\frac{1}{2}\hat{k}_i$. These are combinations of the classical invariants I_1, I_2 .

Our last recurrence relation completely determines K_0 . Indeed, in the real case $K_0 i_3 = i_4 - (i_3)_x$ implies that $K_0 i_3$ has vanishing e_1 component. If K_0 is given as in (5.7), this condition implies $c = b = k_4$ and $a = k_3$. In the complex case, $K_0 i_3 = i_4 - (i_3)_x$ implies also that the e_1 component of $K_0 e_3$ vanishes, which in this case implies $a = 0$. Notice that one can go back in our calculations and obtain an explicit formula for these invariants, if needed. We just proved the following theorem.

Theorem 6. *Let ρ be the left conformal moving frame found in the previous subsection. Then $\rho^{-1} \rho_x = K$ where K is of the form*

$$K = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & k_3 & k_4 & k_4 & 0 & 0 \\ k_1 & k_4 & 0 & 0 & -k_4 & 1 \\ k_2 & k_4 & 0 & 0 & -k_4 & -1 \\ 0 & 0 & -k_4 & -k_4 & -k_3 & 0 \\ 0 & 0 & -k_2 & -k_1 & 0 & 0 \end{pmatrix}$$

in the real case, and

$$K = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ -k_2 & 0 & k_3 & k_3 & 0 & 0 \\ k_1 & k_4 & 0 & 0 & -k_3 & 1 \\ -k_1 & k_4 & 0 & 0 & -k_3 & -1 \\ k_2 & 0 & -k_4 & -k_4 & 0 & 0 \\ 0 & -k_2 & k_1 & -k_1 & k_2 & 0 \end{pmatrix}$$

in the complex case. The invariants k_1 and k_2 generate all differential invariants of third order, while k_3 and k_4 generate independent invariants of higher order. Explicit algebraic formulas can be found for each one of the invariants in terms of derivatives of u .

6. NATURAL FRAMES FOR CONFORMAL AND GRASSMANNIAN CURVES

The concept of a natural frame appeared in [2]. The idea of the author was that, although widely used, classical Serret-Frenet frames in Riemannian geometry were, by no means, the only way to frame Riemannian curves, and often not the most convenient one. The author of the article showed that one could, for example, find a classical moving frame (which he called *natural moving frame*) for which the

derivative of any of the vectors in the frame, other than the unit tangent, will be in the tangential direction (thus, it is often called the *parallel frame*). Physically, the moving frame does not record any rotational movement on the normal plane. That is, if $\nu = (T, T_1, \dots, T_{n-1})$ is a classical Serret-Frenet moving frame, T the unit tangent, then

$$\nu_x = \nu \begin{pmatrix} 0 & -\kappa_1 & 0 & \dots & 0 \\ \kappa_1 & 0 & -\kappa_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \kappa_{n-2} & 0 & -\kappa_{n-1} \\ 0 & \dots & 0 & \kappa_{n-1} & 0 \end{pmatrix}$$

while if $\eta = (T, N_1, \dots, N_{n-1})$ is the natural frame, then

$$\eta_x = \eta \begin{pmatrix} 0 & -k_1 & -k_2 & \dots & -k_{n_1} \\ k_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{n-1} & 0 & 0 & \dots & 0 \end{pmatrix}$$

A main characteristic of this natural frame is that their vectors are *non-local*, that is, they cannot be written as an algebraic expression of derivatives of the curve. In fact, one needs to solve a linear differential equation to find the natural frame.

The concept of Riemannian natural moving frame was translated into the conformal picture in [13]. In the case of the Möbius sphere of signature $(n+1, 1)$ one could find a classical moving frame (T, T_1, \dots, T_{n-1}) using a normalizing Riemannian-type process (see [14]). In this process, and unlike the Riemannian case, T had no role and the role of generating the rest of the frame through differentiation was carried out by T_1 instead of the tangent (T_1 is a vector of differential order 3 related to G_3 in (5.2)). In the conformal case the author of [14] defined the classical *conformal natural moving frame* to be the frame for which the derivatives of all vectors other than T and T_1 are conformally in the direction of T_1 . In terms of the group-based Maurer-Cartan matrix, if ρ_C is the moving frame associated to the classical Serret-Frenet frame (T, T_1, \dots, T_{n-1}) as in [14] and ρ_N is the one associated to (T, N_1, \dots, N_{n-1}) , the natural frame, then

$$(\rho_C)_x = \rho_C \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ k_1 & 0 & 0 & 0 & \dots & 0 & 1 \\ k_2 & 0 & 0 & -k_3 & 0 & \dots & 0 \\ 0 & 0 & k_3 & 0 & -k_4 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & -k_n & 0 \\ 0 & 0 & \dots & 0 & k_n & 0 & 0 \\ 0 & k_1 & k_2 & 0 & \dots & 0 & 0 \end{pmatrix}$$

while

$$(6.1) \quad (\rho_N)_x = \rho_N \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ k_1 & 0 & 0 & 0 & \dots & 0 & 1 \\ k_2 & 0 & 0 & -\kappa_3 & \dots & -\kappa_n & 0 \\ 0 & 0 & \kappa_3 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \kappa_n & 0 & \dots & 0 & 0 \\ 0 & k_1 & k_2 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

As in the Riemannian case, the conformal natural frame is non-local and one needs to solve a linear differential equation to find it. More interestingly, the author of [14] showed that if the geometric Poisson bracket is to be written using a classical Serret-Frenet equation, the Poisson structure does not restrict to the submanifold $k_i = 0$, $i = 3, \dots$, while when written in terms of natural ones the restriction to $\kappa_i = 0$, $i = 3, \dots$, can be carried out to obtain a complexly coupled system of KdV structures. The restriction is biHamiltonian and one indeed obtains a geometric realization of a complexly coupled system of KdV equations for k_1 and k_2 associated to this restriction. Our hope is that, once we find the equivalent definition of natural frame for our conformal and Grassmannian manifolds, we will also be able to obtain restrictions and realizations for both cases.

The definition of natural frames is not well-suited to our algebraic approach of describing moving frames. Therefore, we will try to describe natural frames algebraically in a way that can be readily applied to our situation. Notice that, if one has a left group-based moving frame, the factor of the frame that acts linearly on the manifold (in our case ρ_0) represents a classical frame (see [12]). The map $M \rightarrow M$ taking u to $\rho_0 \cdot u$ is linear and so represented by an element of $\text{GL}(n, \mathbb{R})$. That element has in its columns a classical moving frame. As shown in [13], the way these elements behave under differentiation will be reflected in K_0 , according to the gradation.

Let's first have a good look at the conformal $(n+1, 1)$ case, the normalization constants for the frame ρ_N in (6.1) are $i_1 = (1, \dots, 0)^t$ and $i_3 = (k_1, k_2, 0, \dots, 0)^t$ (they correspond to the K_{-1} and K_1 components of K , respectively). If Θ is the analogous to ours for the $(n+1, 1)$ signature, this choice of constant implies that $\Theta u_1 = \|u_1\|^2 e_1$ and $\Theta G_3 = k_1 e_1 + k_2 e_2$; that is, the first column of the classical moving frame $\alpha^{-1} \Theta^{-1}$ will be the unit tangent and the second a combination of the tangent with G_3 . The fact that all other derivatives are in the direction of the second vector is reflected in the form of K_0 , namely

$$K_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -\kappa_3 & \dots & -\kappa_n \\ 0 & \kappa_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \kappa_n & 0 & \dots & 0 \end{pmatrix}.$$

Algebraically, $K_0 e_i$ is a multiple of e_2 for any $i = 3, \dots$

In our signature $(2, 2)$ conformal sphere $i_1 = e_3$ and, for the real case $i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_2$ and for the complex case $i_3 = \hat{k}_1 e_3 + \hat{k}_2 e_4$. Therefore, in the real case we will say a moving frame is natural if its associated matrix K_0 satisfies $K_0 e_3 = 0$ and $K_0 e_i$ is a multiple of e_2 for $i = 1, 4$. In the complex case, we substitute e_2 with

e_4 . Condition $K_0 e_3 = 0$ implies

$$K_0 = \begin{pmatrix} a & b & b & 0 \\ c & 0 & 0 & -b \\ c & 0 & 0 & -b \\ 0 & -c & -c & -a \end{pmatrix}$$

while

$$K_0 e_1 = \begin{pmatrix} a \\ c-b \\ c-b \\ -a \end{pmatrix}, \quad K_0 e_2 = \begin{pmatrix} 2b \\ 0 \\ 0 \\ -2c \end{pmatrix}, \quad K_0 e_4 = \begin{pmatrix} a \\ c+b \\ c+b \\ a \end{pmatrix}.$$

From here, in the real case we need $a = 0$, while in the complex case we need $b = c$.

Theorem 7. *In the real case, there exists a conformal moving frame ρ_N , one we will call natural moving frame, such that*

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \kappa_3 & \kappa_3 & 0 & 0 \\ k_1 & \kappa_4 & 0 & 0 & -\kappa_3 & 1 \\ \kappa_2 & \kappa_4 & 0 & 0 & -\kappa_3 & -1 \\ 0 & 0 & -\kappa_4 & -\kappa_4 & 0 & 0 \\ 0 & 0 & -k_2 & -k_1 & 0 & 0 \end{pmatrix}.$$

In the complex case, the natural moving frame also exists and it is given by a matrix of the form

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & \kappa_3 & \kappa_4 & \kappa_4 & 0 & 0 \\ k_1 & \kappa_4 & 0 & 0 & -\kappa_4 & 1 \\ \kappa_2 & \kappa_4 & 0 & 0 & -\kappa_4 & -1 \\ 0 & 0 & -\kappa_4 & -\kappa_4 & -\kappa_3 & 0 \\ 0 & 0 & -k_2 & -k_1 & 0 & 0 \end{pmatrix}.$$

Proof. Given any two left moving frames ρ_1 and ρ_2 , $\rho_1 = \rho_2 g$ with g depending on differential invariants ($g = \rho_2^{-1} \rho_1$). If K_1 and K_2 are their two Maurer-Cartan matrices, then they are related by the gauge

$$g^{-1} g_x + g^{-1} K_2 g = K_1.$$

Therefore, we need to find an invariant g gauging K as in Theorem 6 to K_N above. One can directly check that, in the real case, $\rho_N = \rho g$, where

$$g = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma^{-1} \end{pmatrix}$$

and where γ is the solution to the differential equation $\gamma_x = -k_3 \gamma$, that is, $\gamma = e^{-\int k_3}$. The invariants k_3 and k_4 change accordingly into κ_3 and κ_4 .

In the complex case it is only slightly more complicated:

$$g = \begin{pmatrix} \gamma & \eta & \eta & \gamma - 1 \\ -\eta & \gamma & \gamma - 1 & -\eta \\ -\eta & \gamma - 1 & \gamma & -\eta \\ \gamma - 1 & \eta & \eta & \gamma \end{pmatrix}$$

with $1 + \frac{\gamma}{\eta} \frac{\gamma-1}{\eta} = 0$ and $\frac{\gamma}{\eta} = -\tan\left(\frac{1}{2} \int (k_3 + k_4)\right)$. Again, the invariants k_4 and k_3 transform accordingly. \square

These two transformations $k_3, k_4 \rightarrow \kappa_3, \kappa_4$ are generalizations of the well-known *Hasimoto transformation* for Euclidean geometry to the conformal case. The Hasimoto transformation was proven to be a map from classical Frenet to natural moving frames (see [11]).

These moving frames can be translated into the Grassmannian picture using the isomorphism. For completeness we will describe natural frames and how to find them in the Grassmannian picture also. Notice that, even though in the conformal case one can think of geometric ways to define natural frames, this is far less intuitive and not at all obvious in the Grassmannian manifold. It is an interesting question whether or not this can be done for any flat parabolic manifold.

Theorem 8. *In the real case, there exists a left Grassmannian moving frame (the natural frame) such its Maurer-Cartan matrix is given by*

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} 0 & \kappa_3 & 1 & 0 \\ \kappa_4 & 0 & 0 & 1 \\ k_1 & 0 & 0 & \kappa_3 \\ 0 & k_2 & \kappa_4 & 0 \end{pmatrix}.$$

In the complex case, the natural moving frame also exists and it satisfies

$$K_N = \rho_N^{-1}(\rho_N)_x = \begin{pmatrix} \kappa_3 & \kappa_4 & 1 & 0 \\ \kappa_4 & -\kappa_3 & 0 & 1 \\ k_1 & -k_2 & \kappa_3 & \kappa_4 \\ k_2 & k_1 & \kappa_4 & -\kappa_3 \end{pmatrix}.$$

We do not need to prove this theorem, it is true due to the existence of the isomorphism. Still, we can explicitly find the relation to the moving frame we previously found. In the real case, $\rho_N = \rho g$ where

$$g = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma^{-1} & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma^{-1} \end{pmatrix}$$

with $\gamma = e^{-\int k_4}$. In the complex case

$$g = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}$$

where $\theta = -\int k_4$.

7. INVARIANT EVOLUTIONS AND BI-HAMILTONIAN EQUATIONS IN TERMS OF NATURAL FRAMES: COUPLED AND DECOUPLED SYSTEMS OF KdV EQUATIONS

Finally, in this section we will show that both Hamiltonian structures (4.1) and (4.2) reduce to the submanifold of \mathcal{K} defined by $\kappa_3 = \kappa_4 = 0$ to produce a bi-Hamiltonian decoupled system of two KdV structures in the real case, and a bi-Hamiltonian complexly coupled system of KdV equations in the complex case. This

result also implies the existence of geometric realizations for these completely integrable systems. These geometric realizations are best understood as a limit case: as $\kappa_3, \kappa_4 \rightarrow 0$, the evolution of k_1 and k_2 become completely integrable. Indeed, κ_3, κ_4 , in principle, cannot geometrically vanish, as seen in the previous definition of natural frame (in the real case, k_3 will need to blow up for κ_3 to vanish). This is a surprising geometric realization of KdV systems in the Grassmannian case, but it also shows an unexpected result, namely the existence of geometric realizations of decoupled systems of KdV equations in the conformal case. Only the one for complexly coupled systems of KdV was previously known. This realization can be readily found in a general conformal sphere. For this, we merely need to choose the appropriate normalization i_3 as in (5.5) in the work found in [14]. That leads to the appropriate K_1 -form for the matrix.

From now on we will assume that we are working with the natural Maurer-Cartan matrices. To describe the reduction and the reduced brackets we will follow section 4. We will also assume that we are identifying the dual to the Lie algebra with the Lie algebra itself using the trace. That means that the dual to the entry a_{ij} is the entry a_{ji} . In this section both the reduction of the structures, and their relation to geometric realization are explained. Assume we have a Hamiltonian functional $f : \mathcal{K} \rightarrow \mathbb{R}$ defined on the space of Grassmannian differential invariants, i.e., in the space $C^\infty(S^1) \times C^\infty(S^1) \times C^\infty(S^1) \times C^\infty(S^1)$. Assume \mathcal{F} is an extension which is constant on the leaves of the subgroup N , and assume

$$(7.1) \quad \frac{\partial \mathcal{F}}{\partial L}(K) = \begin{pmatrix} F_2 & F_{-1} \\ F_1 & F_3 \end{pmatrix}, \quad K = \begin{pmatrix} K_2 & I \\ K_1 & K_2 \end{pmatrix}$$

Real case. In this case $K_1 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ and $K_2 = \begin{pmatrix} 0 & \kappa_3 \\ \kappa_4 & 0 \end{pmatrix}$. If we denote $f_i = \frac{\partial f}{\partial k_i}(\mathbf{k})$, then,

$$(7.2) \quad F_{-1} = \begin{pmatrix} f_1 & \alpha \\ \beta & f_2 \end{pmatrix}, F_2 = \begin{pmatrix} a & \frac{1}{2}f_4 + f \\ \frac{1}{2}f_3 + e & b \end{pmatrix}, F_3 = \begin{pmatrix} c & \frac{1}{2}f_4 - f \\ \frac{1}{2}f_3 - e & -a - b - c \end{pmatrix}$$

and F_1 is arbitrary. In this case, the parabolic subgroup P_G is formed by matrices of the block form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Therefore, $\mathfrak{n}^0 = \mathfrak{p}_G^0$ can be identified with matrices with block-form

$$\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

Substituting (7.1) in (4.3) we get

$$(7.3) \quad F'_{-1} + F_3 - F_2 + [K_2, F_{-1}] = 0,$$

$$(7.4) \quad F'_2 + [K_2, F_2] - F_{-1}K_1 + F_1 = 0$$

$$(7.5) \quad F'_3 + [K_3, F_3] + K_1F_{-1} - F_1 = 0$$

We can readily solve for $F_2 - F_3$ using (7.3) and for F_1 using (7.4) minus (7.5). The rest of the entries are uniquely determined by the entries in which f'_i are located

in the equations. Solving we get

$$\begin{aligned}
 \alpha &= \frac{1}{k_2 - k_1} (f'_4 - 2\kappa_3 D^{-1}(\kappa_4 f_4 - \kappa_3 f_3)) \\
 \beta &= \frac{1}{k_2 - k_1} (f'_3 + 2\kappa_4 D^{-1}(\kappa_4 f_4 - \kappa_3 f_3)) \\
 2a &= f'_1 + \kappa_3 \beta - \alpha \kappa_4 + D^{-1}(\kappa_4 f_4 - \kappa_3 f_3) \\
 2c &= -f'_1 - \kappa_3 \beta + \alpha \kappa_4 + D^{-1}(\kappa_4 f_4 - \kappa_3 f_3) \\
 2b &= f'_2 - \kappa_3 \beta + \alpha \kappa_4 - D^{-1}(\kappa_4 f_4 - \kappa_3 f_3) \\
 2f &= \alpha' + \kappa_3 (f_2 - f_1) \\
 2e &= \beta' + \kappa_4 (h_1 - h_2)
 \end{aligned}$$

Using this results we can calculate the reduction of (4.1) to the space of differential invariants. If f and h are two such functionals, then the Geometric Hamiltonian structure is defined as

$$\{f, h\}(\mathbf{k}) = \int_{S^1} \text{trace} \left(\left(\frac{\delta \mathcal{F}}{\delta L}(K) \right)' + \left[K, \frac{\delta \mathcal{F}}{\delta L}(K) \right] \right) \frac{\delta \mathcal{H}}{\delta L}(K) dx$$

where $\frac{\delta \mathcal{H}}{\delta L}(K)$ is calculated similarly to the variational derivative of \mathcal{F} . The author of [12] proved that this bracket is always a Poisson bracket. To show that the bracket can be further restricted to the submanifold $\kappa_3 = \kappa_4 = 0$ we will check that, if f depends only on k_1 and k_2 (that is, $f_3 = f_4 = 0$), while h depends only on κ_3 and κ_4 (that is, $h_2 = h_1 = 0$), then $\{f, h\}(\mathbf{k}) = 0$ along the submanifold $\kappa_3 = \kappa_4 = 0$. After this step, we can calculate the bracket of two functionals that depend only on k_1 and k_2 .

First of all, notice that condition (4.3) implies that

$$\{f, h\}(\mathbf{k}) = \int_{S^1} \text{trace} ((F'_1 + K_1 F_2 - F_3 K_1) H_{-1}) dx.$$

If a functional f satisfies $f_3 = f_4 = 0$, then we have $\alpha = \beta = e = f = 0$ and $a = \frac{1}{2}f'_1 + C$, $b = \frac{1}{2}f'_2 - C$, $c = -\frac{1}{2}f'_1 + C$, where C is a possible constant that will end up canceling out in the calculation of the bracket. These values imply that, whenever $\kappa_3 = \kappa_4 = 0$, the matrix $F'_1 + K_1 F_2 - F_3 K_1$ will be diagonal. On the other hand, if $h_1 = h_2 = 0$, then H_{-1} will have a vanishing diagonal and, therefore, $\{f, h\}(\mathbf{k}) = 0$.

Finally, if both f and h depend on k_1, k_2 only, then, when $\kappa_3 = \kappa_4 = 0$

$$F'_1 + K_1 F_2 - F_3 K_1 = \begin{pmatrix} -\frac{1}{2}f''_1 + k_1 f_1 & 0 \\ 0 & -\frac{1}{2}f''_2 + k_2 f_2 \end{pmatrix}_x + \begin{pmatrix} k_1 f'_1 & 0 \\ 0 & k_2 f'_2 \end{pmatrix},$$

and therefore

$$\{f, h\}(\mathbf{k}) = \int_{S^1} \text{trace} (F'_1 + K_1 F_2 - F_3 K_1) H_{-1} dx = \int_{S^1} \begin{pmatrix} \frac{\delta f}{\delta k_1} & \frac{\delta f}{\delta k_2} \end{pmatrix} \mathcal{P} \begin{pmatrix} \frac{\delta h}{\delta k_1} \\ \frac{\delta h}{\delta k_2} \end{pmatrix} dx$$

where

$$\mathcal{P} = \begin{pmatrix} -\frac{1}{2}D^3 + k_1 D + Dk_1 & 0 \\ 0 & -\frac{1}{2}D^3 + k_2 D + Dk_2 \end{pmatrix}$$

which defines a decoupled Hamiltonian structure for KdV. One can also check directly that, if we substitute the values of (7.1) in (4.2) for the choice

$$L_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

the resulting bracket is given by

$$\{f, h\}_0(k_1, k_2) = \frac{1}{2} \int_{S^1} \begin{pmatrix} \frac{\delta f}{\delta k_1} & \frac{\delta f}{\delta k_2} \end{pmatrix} \mathcal{P}_0 \begin{pmatrix} \frac{\delta h}{\delta k_1} \\ \frac{\delta h}{\delta k_2} \end{pmatrix} dx$$

where \mathcal{P}_0 is the second Hamiltonian structure for a decoupled system of KdV equations, that is

$$\mathcal{P}_0 = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}.$$

One can also check that the bracket of two Hamiltonians depending on (k_1, k_2) only and (κ_3, κ_4) only vanishes when κ_3 and κ_4 do.

Complex case. In the complex case the same procedure is followed but with different matrices $K_1, K_2, F_1, F_{-1}, F_2$ and F_3 . In this case

$$K_1 = \begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \kappa_3 & \kappa_4 \\ -\kappa_3 & \kappa_4 \end{pmatrix}$$

and

$$F_{-1} = \begin{pmatrix} \frac{1}{2}f_1 + \alpha & \frac{1}{2}f_2 + \beta \\ -\frac{1}{2}f_2 + \beta & \frac{1}{2}f_1 - \alpha \end{pmatrix}$$

$$F_2 = \begin{pmatrix} \frac{1}{4}f_3 + a & \frac{1}{4}f_4 + f \\ \frac{1}{4}f_4 + b & -\frac{1}{4}f_3 + c \end{pmatrix}, \quad F_3 = \begin{pmatrix} \frac{1}{4}f_3 - a & \frac{1}{4}f_4 + e \\ \frac{1}{4}f_4 + d & -\frac{1}{4}f_3 - c \end{pmatrix}.$$

In this case, condition (4.3) has the same equations as the real case, but this result in different values for our unknowns

$$\begin{aligned} \alpha &= -\frac{1}{4k_2}f'_4 + \frac{\kappa_3}{k_2}D^{-1}(\kappa_3f_4 - \kappa_4f_3) \\ \beta &= \frac{1}{4k_2}f'_3 + \frac{\kappa_4}{k_2}D^{-1}(\kappa_3f_4 - \kappa_4f_3) \\ 2a &= \alpha' + \frac{1}{2}f'_1 - \kappa_4f_2 \\ 2c &= -\alpha' + \frac{1}{2}f'_1 + \kappa_4f_2 \\ b + d &= D^{-1}(\kappa_3f_4 - \kappa_4f_3) \\ b - d &= \beta' - \frac{1}{2}f'_2 + \kappa_3(f_2 - 2\beta) + 2\kappa_4\alpha \\ f + e &= -D^{-1}(\kappa_3f_4 - \kappa_4f_3) \\ f - e &= \beta' + \frac{1}{2}f'_2 + \kappa_3(f_2 + 2\beta) - 2\kappa_4\alpha. \end{aligned}$$

As before, if $f_3 = f_4 = 0$ and $h_1 = h_2 = 0$, one can check straightforward that $\{f, h\}(k_1, k_2) = 0$. The calculations are only slightly longer than the ones for the real case. Also, if both f and h satisfy $f_3 = f_4 = h_3 = h_4 = 0$, then

$$\begin{aligned} &H'_1 + K_1H_2 - H_3K_1 \\ &= \frac{1}{2} \begin{pmatrix} -\frac{1}{2}f'''_1 + (k_1f_1)' + k_1f'_1 + (k_2f_2)' + k_2f'_2 & -\frac{1}{2}f'''_2 + (k_1f_2)' + k_1f'_2 - (k_2f_1)' - k_2f'_1 \\ \frac{1}{2}f'''_2 - (k_1f_2)' - k_1f'_2 + (k_2f_1)' + k_2f'_1 & -\frac{1}{2}f'''_1 + (k_1f_1)' + k_1f'_1 + (k_2f_2)' + k_2f'_2 \end{pmatrix}. \end{aligned}$$

Putting all of these values together in (4.1) we obtain the restricted bracket to be

$$\{f, h\}(k_1, k_2) = \int_{S^1} \begin{pmatrix} \frac{\delta f}{\delta k_1} & \frac{\delta f}{\delta k_2} \end{pmatrix} \mathcal{P} \begin{pmatrix} \frac{\delta h}{\delta k_1} \\ \frac{\delta h}{\delta k_2} \end{pmatrix} dx$$

where

$$\mathcal{P} = \frac{1}{2} \begin{pmatrix} -\frac{1}{2}D^3 + Dk_1 + k_1D & Dk_2 + k_2D \\ Dk_2 + k_2D & \frac{1}{2}D^3 - Dk_1 - k_1D \end{pmatrix}$$

the well-known Hamiltonian structure for a complexly coupled system of KdV equations. This structure had been already obtained in ???. Accordingly, we can guess that the choice of third normalization in [14] is in a different prolonged orbit that the real case we found here, and that a choice in the same orbit will result in a decoupled system of KdV equations. Once more, if we substitute our values in (4.2) for the same choice of L_0 as in the real case, we obtain a second Poisson structure \mathcal{P}_0 , namely

$$\mathcal{P}_0 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}.$$

This is known to be the second Hamiltonian structure for a complexly coupled KdV system. As before, one can also check that the bracket of Hamiltonians depending on (k_1, k_2) only and (κ_3, κ_4) only vanishes when κ_3 and κ_4 do.

The interest of these different structures is that (4.1) reduces always to a Geometric Poisson bracket that is directly linked to a geometric realization of the Hamiltonian evolution. The relation is given as in (4.5). Therefore, we have almost finished the proof of the following Theorem.

Theorem 9. *Assume $u(t, x)$ describes an evolution of a curve of Grassmannian planes in \mathbb{R}^4 solution of the equation*

$$(7.6) \quad u_t = u_1 S(u) = u_3 - \frac{3}{2} u_2 u_1^{-1} u_2.$$

Then, if $S(u)$ has real eigenvalues k_1 and k_2 , as $\kappa_3, \kappa_4 \rightarrow 0$, the curvatures k_1 and k_2 satisfy a decoupled system of KdV equations. If $S(u)$ has complex eigenvalues, $k_1 \pm k_2 i$, then k_1 and k_2 satisfy a complexly coupled system of KdV equations. In both cases, the Poisson structures (4.1) and (4.2) reduce to the space $\kappa_3 = \kappa_4 = 0$ to produce a biHamiltonian pencil for decoupled KdVs or complexly coupled KdVs, depending on the case.

Proof. Assume $(\frac{\delta \mathcal{H}}{\delta L}(K))_{-1} = g_r$, with $g_r = \begin{pmatrix} 0 & \mathbf{r} \\ 0 & 0 \end{pmatrix}$, \mathbf{r} a matrix of differential invariants, and where \mathcal{H} is the appropriate extension of a Hamiltonian functional h , and where by $(\)_{-1}$ we indicate the projection on the tangent to the manifold G/P , as represented by the subspace \mathfrak{g}_{-1} of the Lie algebra \mathfrak{g} . Then, from [12], the evolution

$$\rho_{-1}^{-1}(\rho_{-1})_t = Ad(\rho_0)g_r$$

induces the h -hamiltonian evolution on the differential invariants. If we choose $h(k_1, k_2) = \frac{1}{2} \int_{S^1} (k_1^2 + k_2^2) dx$, then $h_1 = k_1$ and $h_2 = k_2$ and $H_{-1} = \mathbf{r} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$.

Calculating (4.5) with the natural moving frame we obtain

$$\begin{pmatrix} I & -u \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & u_t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u_1 B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{r} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B u_1^{-1} & 0 \\ 0 & B \end{pmatrix}$$

which results in

$$u_t = u_1 B^{-1} \mathbf{r} B.$$

If we now look at the third normalization equations, notice that i_3 is either $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ in the real case, or $\begin{pmatrix} k_1 & -k_2 \\ k_2 & k_1 \end{pmatrix}$ in the complex case. In either situation, if we choose $\mathbf{r} = i_3$, then we get $B^{-1} i_3 B = S(u)$, since the third normalization equation in both cases read $BS(u)B^{-1} = i_3$.

The choice of h clearly results in the completely integrable systems the Theorem states. \square

Notice that this immediately implies that equation (7.6) preserves the level set $\kappa_3 = \kappa_4 = 0$, a fact that can also be found directly using the techniques we used in section 3.3.

The last results in this paper are proven choosing a different, but also non-local, moving frame. This moving frame corresponds to the Laguerre-Forsyth's canonical form of the Serret-Frenet equations for ρ , and it was linked to Grassmannians in [18]. The invariants generated by this last Grassmannian moving frame evolve following a noncommutative KdV equation. When reduced to these invariants, both brackets (4.1) and (4.2) become the biHamiltonian structure for noncommutative KdV.

Theorem 10. *There exists a (non-local) Grassmannian moving frame such that its associated Maurer-Cartan matrix is given by*

$$\begin{pmatrix} 0 & I \\ \widehat{K} & 0 \end{pmatrix}$$

where the entries of \widehat{K} are independent and generating differential invariants. Both structures (4.1) and (4.2) (with the choice $L_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$) reduce to \mathcal{K} as represented by these invariants to produce a biHamiltonian structure for the noncommutative KdV equation. Furthermore, the evolution

$$u_t = u_1 \widehat{K}$$

induces a noncommutative KdV equation on \widehat{K} .

Proof. Since the technique is identical to the previous cases, we will describe the first part of the calculations without too many explanations. If we gauge (3.17) or (3.18) by an element of the form

$$g = \begin{pmatrix} \Theta & 0 \\ 0 & \Theta \end{pmatrix}$$

the result is

$$\begin{pmatrix} \Theta^{-1}\Theta_x + \Theta^{-1}K_0\Theta & I \\ \Theta^{-1}K_1\Theta & \Theta^{-1}\Theta_x + \Theta^{-1}K_0\Theta \end{pmatrix}.$$

Therefore, choosing Θ to be the solution of $\Theta_x = -K_0\Theta$ results on the choice $\widehat{K} = \Theta^{-1}K_1\Theta$. Notice that the invariants k_1 and k_2 are generated by the trace and determinant of \widehat{K} .

We now calculate the reduction of the Poisson brackets. Assume

$$\frac{\delta\mathcal{H}}{\delta L} = \begin{pmatrix} A & \mathbf{h} \\ C & B \end{pmatrix}$$

is the derivative of an appropriate extension of h with $\mathbf{h} = \frac{\delta h}{\delta \kappa}$. Then, equation (4.3) results in the values

$$\begin{aligned} A &= \frac{1}{2}\mathbf{h}_x - \frac{1}{2}D^{-1}(\widehat{K}\mathbf{h} - \mathbf{h}\widehat{K}) \\ B &= -\frac{1}{2}\mathbf{h}_x - \frac{1}{2}D^{-1}(\widehat{K}\mathbf{h} - \mathbf{h}\widehat{K}) \\ C &= -\frac{1}{2}\mathbf{h}_{xx} + \frac{1}{2}(\widehat{K}\mathbf{h} + \mathbf{h}\widehat{K}). \end{aligned}$$

Using this extension, the reduction of (4.1) is given by

$$\{h, f\}_{R(\kappa)} = \int \text{tr} \left((C_x + \widehat{K}A - B\widehat{K}) \frac{\delta f}{\delta \kappa} \right) dx = \int \mathbf{h}^T \mathcal{P} \mathbf{f} dx$$

where

$$2\mathcal{P}\mathbf{h} = -\mathbf{h}_{xxx} + (\mathbf{h}\widehat{K} + \widehat{K}\mathbf{h})_x + \widehat{K}\mathbf{h}_x + \mathbf{h}_x\widehat{K} - \widehat{K}D^{-1}(\widehat{K}\mathbf{h} - \mathbf{h}\widehat{K}) + D^{-1}(\widehat{K}\mathbf{h} - \mathbf{h}\widehat{K})\widehat{K}.$$

Choosing $L_0 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$ in (4.2) and using the expression for the extension $\frac{\delta \mathcal{H}}{\delta L}$, we have that the reduction of (4.2) is given by

$$\{h, f\}_0(\kappa) = \int \mathbf{h}^T D \mathbf{f} dx.$$

These two structure are the ones appearing in [16] as biHamiltonian structures for noncommutative KdV. The calculation of the geometric realization is identical to the previous cases.

There is one last point that need to be checked. An arbitrary gauge by a matrix of invariants of a moving frame it does result in a new moving frame, but the entries of its Maurer-Cartan matrix do not necessarily generate all other invariants. Indeed, the proof in [7] shows that if a moving frame is found using normalization constants (as in the first frame we found), then they do indeed generate. But, although perhaps true, a general theorem for any moving frame has not been proved yet. Therefore, we need to check that such is the case here.

We know that k_1 and k_2 are generated by the entries of \widehat{K} . Lengthy but straightforward calculations show that, if $\Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\widehat{K} = \begin{pmatrix} \widehat{\kappa}_1 & \widehat{\kappa}_2 \\ \widehat{\kappa}_3 & \widehat{\kappa}_4 \end{pmatrix}$, then

$$\frac{a'a}{bb'} = -\frac{\widehat{\kappa}_3}{\widehat{\kappa}_2}, \quad \frac{a'b}{ab'} = \frac{k_2^2 - k_1^2}{\widehat{\kappa}_1 k_1 - \widehat{\kappa}_4 k_2} + 1.$$

This implies that $n_1 = \frac{a'}{b'}$ and $n_2 = \frac{a}{b}$ are functionally generated by $\widehat{\kappa}_i$, $i = 1, 2, 3, 4$. Further calculations show that $\kappa_3 \kappa_4 = -n_1' n_2'$ and

$$\frac{\kappa_3'}{\kappa_3} n_2' - 2 \frac{n_2'}{n_1 - n_2} = n_2''$$

concluding that both κ_3 and κ_4 are also formally functionally generated by the entries of \widehat{K} . □

Our final corollaries state the existence of a conformal level set for a decoupled system of KdV equations. It also shows that the signature $(2, 2)$ conformal Poisson brackets are equivalent to the bi-Hamiltonian structure for non-commutative KdV

2×2 equations. Although this is immediate under the isomorphism, it was not previously obvious.

Corollary 1. *There exists a conformally invariant evolution of curves (written in terms of a natural classical moving frame) such that it preserves $\kappa_3 = \kappa_4 = 0$. As $\kappa_3, \kappa_4 \rightarrow 0$, the evolution induces a decoupled system of KdV equations on k_1 and k_2 .*

Corollary 2. *Both brackets (4.1) and (4.2) reduce to the space of conformal differential invariants to produce a biHamiltonian structure equivalent to that of the non-commutative KdV equation that appears in [16].*

The interested reader can find the exact equations for these geometric realizations using the isomorphism with the corresponding Grassmannian evolution.

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