

Math 221-Midterm 1 *Solutions*

Name: _____

Your TA: (circle one)

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Directions—Please Read Carefully!

- There are 7 problems and 8 pages.
- Calculators and extraneous scratch paper are not allowed.
- Show all your work. Justify your answers and write clearly. Unsubstantiated/illegible answers will receive little or no credit.
- If more space is needed, use the back of the previous page to continue your work. Be sure to make a note of this on the problem page so that the grader knows where to find your answers. Raise your hand if more scratch paper is needed.
- *Good Luck!!!*

Problem Number	Possible Points	Score
1	18	
2	12	
3	12	
4	10	
5	20	
6	10	
7	18	
Total	100	

1. (18 points) Calculate the derivatives of the following functions by using differentiation rules.

(a) $y = \frac{x}{x^3 + 4}$ (please simplify your answer)

$$y' = \frac{\left(\frac{d}{dx} x\right)(x^3 + 4) - x \frac{d}{dx}(x^3 + 4)}{(x^3 + 4)^2} = \frac{x^3 + 4 - x \cdot 3x^2}{(x^3 + 4)^2} = \frac{4 - 2x^3}{(x^3 + 4)^2}$$

(b) $y = x \sin(\sqrt[3]{x} + 1)$ (You DO NOT need to simplify your answer)

$$\begin{aligned} y' &= \sin(\sqrt[3]{x} + 1) + x \cos(\sqrt[3]{x} + 1) \cdot \frac{d}{dx}(\sqrt[3]{x} + 1) \\ &= \sin(\sqrt[3]{x} + 1) + x \cos(\sqrt[3]{x} + 1) \cdot \left(\frac{1}{3} x^{-\frac{2}{3}}\right) \end{aligned}$$

(c) $y = \left(\frac{1}{x^2} + \cos^2 x\right)^7$ (You DO NOT need to simplify your answer)

$$\begin{aligned} y' &= 7 \left(\frac{1}{x^2} + \cos^2 x\right)^6 \cdot \frac{d}{dx} \left(\frac{1}{x^2} + \cos^2 x\right) \\ &= 7 \left(\frac{1}{x^2} + \cos^2 x\right)^6 \cdot [-2x^{-3} + 2\cos x \cdot (-\sin x)] \end{aligned}$$

2. (12 points)

(a) Compute the left-hand and the right-hand limits of $\frac{x^2+x}{|x+1|}$ at -1 and determine whether

$\lim_{x \rightarrow -1} \frac{x^2+x}{|x+1|}$ exists.

$$|x+1| = \begin{cases} x+1 & \text{if } x \geq -1 \\ -(x+1) & \text{if } x < -1 \end{cases}$$

$$\lim_{x \rightarrow -1^+} \frac{x^2+x}{|x+1|} = \lim_{x \rightarrow -1^+} \frac{x^2+x}{x+1} = \lim_{x \rightarrow -1^+} \frac{x(x+1)}{x+1} = \lim_{x \rightarrow -1^+} x = -1$$

$$\lim_{x \rightarrow -1^-} \frac{x^2+x}{|x+1|} = \lim_{x \rightarrow -1^-} \frac{x^2+x}{-(x+1)} = \lim_{x \rightarrow -1^-} \frac{x(x+1)}{-(x+1)} = \lim_{x \rightarrow -1^-} (-x) = 1$$

Since $\lim_{x \rightarrow -1^+} \frac{x^2+x}{|x+1|} \neq \lim_{x \rightarrow -1^-} \frac{x^2+x}{|x+1|}$, the two sided limit $\lim_{x \rightarrow -1} \frac{x^2+x}{|x+1|}$ does not exist.

(b) Compute $\lim_{x \rightarrow 0} x^6 \cos\left(\frac{1}{x}\right)$ by using the Sandwich Theorem.

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad (\text{for all } x \neq 0)$$

$$\Rightarrow -x^6 \leq x^6 \cos\left(\frac{1}{x}\right) \leq x^6 \quad (\text{for all } x \neq 0)$$

Since $\lim_{x \rightarrow 0} x^6 = 0$, $\lim_{x \rightarrow 0} (-x^6) = 0$, by Sandwich thm, $\lim_{x \rightarrow 0} x^6 \cos\left(\frac{1}{x}\right) = 0$

3. (12 points) For each of the problems, circle the (unique) correct answer.

(a) Only one of the following limits exists. Which one?

A. $\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right)$

B. $\lim_{x \rightarrow \infty} \sin x$

C. $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

D. $\lim_{x \rightarrow 0} \frac{1}{\sin x}$

as $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0$ so $\sin\left(\frac{1}{x}\right) \rightarrow \sin(0) = 0$ as $x \rightarrow \infty$.



(b) Suppose $f(x)$ is an **odd**¹ function and the **right-hand limit** at 1 equals 2, i.e. $\lim_{x \rightarrow 1+} f(x) = 2$.

What can we say about the **right-hand limit** at -1, i.e. $\lim_{x \rightarrow -1+} f(x)$?

A. $\lim_{x \rightarrow -1+} f(x) = 2$

B. $\lim_{x \rightarrow -1+} f(x) = -2$

C. There is not enough information to tell what $\lim_{x \rightarrow -1+} f(x)$ is.

If $f(x)$ is odd,

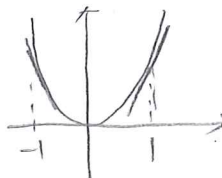
$\lim_{x \rightarrow 1+} f(x) = 2$ implies $\lim_{x \rightarrow -1-} f(x) = -2$. But we can't say anything about $\lim_{x \rightarrow -1+} f(x)$.

(c) Suppose $f(x)$ is an **even**² function and $f'(1) = 3$. What can we say about $f'(-1)$?

A. $f'(-1) = 3$

B. $f'(-1) = -3$ (by symmetry)

C. There is not enough information to tell what $f'(-1)$ is.



(d) Only one of the following statements is correct. Which one?

A. If $\lim_{x \rightarrow 0} f(x) = 1$, then $f(0)$ must be 1.

B. If $f(x)$ is a function defined over the interval $[0, 1]$ with $f(0) < 0$ and $f(1) > 0$, then there must be a point c in $[0, 1]$ such that $f(c) = 0$.

C. If $f(x)$ and $g(x)$ are functions defined near $x = 1$ and $\lim_{x \rightarrow 1} f(x) = 0$, then $\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = 0$.

D. If $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x}$ exists, then $\lim_{x \rightarrow 0} f(x) = f(0)$.

" $f(x)$ is differentiable at 0" implies " $f(x)$ is continuous at 0"

¹A function $f(x)$ is odd if $f(-x) = -f(x)$ for all x in its domain. The graph of an odd function is symmetric with respect to the origin.

²A function $f(x)$ is even if $f(-x) = f(x)$ for all x in its domain. The graph of an even function is symmetric across the y -axis.

4. (10 points) Use the definition of the derivative as a limit (you MUST use this) to find the derivative of $f(x) = \sqrt{x^2 + 3}$ at $x = 1$ and the equation of the tangent line to the graph of $f(x)$ at $x = 1$.

$$\begin{aligned}
 f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(1+h)^2 + 3} - \sqrt{1+3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(\sqrt{(1+h)^2 + 3} - \sqrt{1+3})(\sqrt{(1+h)^2 + 3} + \sqrt{1+3})}{h(\sqrt{(1+h)^2 + 3} + \sqrt{1+3})} \\
 &= \lim_{h \rightarrow 0} \frac{(1+h)^2 + 3 - (1+3)}{h(\sqrt{(1+h)^2 + 3} + \sqrt{1+3})} \\
 &= \lim_{h \rightarrow 0} \frac{2h + h^2}{h(\sqrt{(1+h)^2 + 3} + \sqrt{1+3})} \\
 &= \lim_{h \rightarrow 0} \frac{2+h}{\sqrt{(1+h)^2 + 3} + \sqrt{1+3}} = \frac{2}{\sqrt{1+3} + \sqrt{1+3}} = \frac{2}{4} = \frac{1}{2}
 \end{aligned}$$

$$f(1) = \sqrt{1+3} = 2 \Rightarrow (1, 2) \text{ is a point on the tangent line}$$

$\Rightarrow$ the equation of tangent at $x=1$ is

$$y = \frac{1}{2}(x-1) + 2$$

5. (20 points) Consider the function $f(x) = \frac{x|x|}{x^2 - 4}$.

(a) Find all the horizontal asymptotes of the graph of $f(x)$. Show your work.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2}{x^2 - 4} = \lim_{x \rightarrow \infty} \frac{1}{1 - \frac{4}{x^2}} = 1 \Rightarrow y = 1 \text{ is a horizontal asymptote}$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{-x^2}{x^2 - 4} = \lim_{x \rightarrow -\infty} \frac{-1}{1 - \frac{4}{x^2}} = -1 \Rightarrow y = -1 \text{ is a horizontal asymptote}$$

(b) Find all the vertical asymptotes of the graph of $f(x)$. Show your work.

$f(x) = \frac{x|x|}{(x-2)(x+2)}$ is undefined at $x=2$ and $x=-2$.
possible vertical asymptotes: $x=2$, $x=-2$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{x|x| \rightarrow 4}{\underbrace{(x-2)}_{\rightarrow 0^+} \underbrace{(x+2)}_{\rightarrow 4}} = \infty \Rightarrow x=2 \text{ is a vertical asymptote}$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x|x| \rightarrow 4}{\underbrace{(x-2)}_{\rightarrow 0^-} \underbrace{(x+2)}_{\rightarrow 4}} = -\infty$$

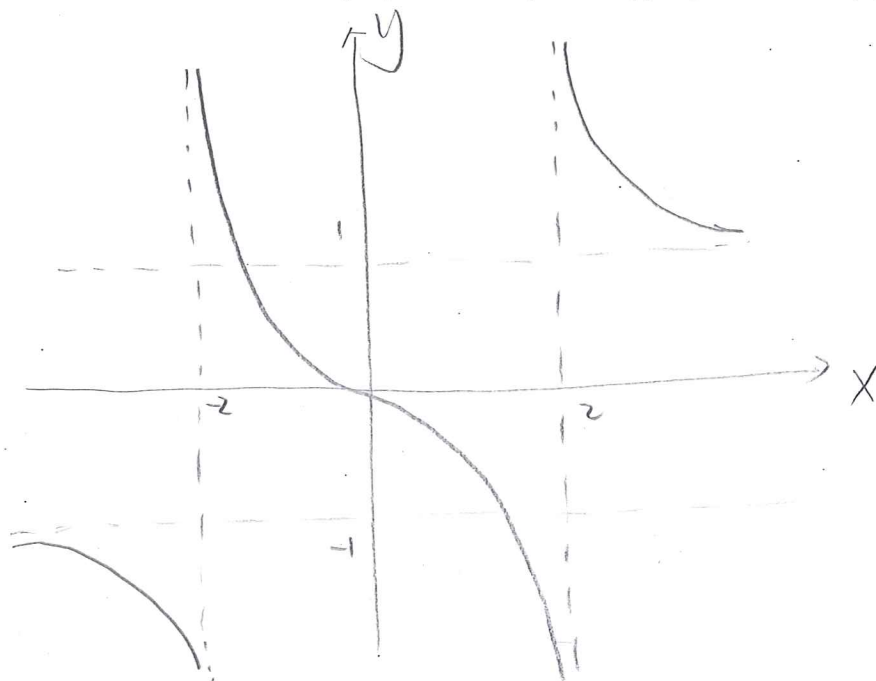
$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \frac{x|x| \rightarrow -4}{\underbrace{(x-2)}_{\rightarrow -4} \underbrace{(x+2)}_{\rightarrow 0^+}} = \infty$$

$$\Rightarrow x = -2 \text{ is a vertical asymptote}$$

$$\lim_{x \rightarrow -2^-} f(x) = -\infty$$

Remark You need to compute one of $\lim_{x \rightarrow 2^+}$ and $\lim_{x \rightarrow -2^-}$ to show $x=2$ is a vertical asymptote
You need to compute both to graph $f(x)$

- (c) Based on your answers to the (a) and (b), graph $f(x)$ as accurately as possible. Be sure to include all horizontal and vertical asymptotes and positivity/negativity of $f(x)$.



6. (10 points) Show that the equation $\cos x + 4x = 0$ has at least one solution by using the Intermediate Value Theorem.

Let $f(x) = \cos x + 4x$. $f(x)$ is continuous for all values of x .

$$f(0) = \cos 0 + 4 \cdot 0 = 1 > 0$$

$$f(-\frac{\pi}{2}) = \cos(-\frac{\pi}{2}) + 4 \cdot (-\frac{\pi}{2}) = 0 - 2\pi < 0$$

Since $f(x)$ is continuous over $[-\frac{\pi}{2}, 0]$, by IVT, there

must exist c in $[-\frac{\pi}{2}, 0]$ such that $f(c) = 0$.

i.e. $\cos x + 4x = 0$ has at least one solution in $[-\frac{\pi}{2}, 0]$

7. (18 points) Consider the function $f(x)$ defined by $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ k(x-1)+1 & \text{if } x > 1 \end{cases}$ where k is a constant and answer the following questions.

(a) For what value (or values) of k is $f(x)$ continuous at $x = 1$? Justify your answer.

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} k(x-1)+1 = k(1-1)+1 = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1$$

$$f(1) = 1^2 = 1$$

Since $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1)$ regardless of what k is,

$f(x)$ is continuous at $x=1$ for all values of k .

(b) For what value (or values) of k is $f(x)$ differentiable at $x = 1$? Justify your answer.

The left-hand derivative of $f(x)$ at $x=1$ is

$$\lim_{h \rightarrow 0^-} \frac{f(1+h)-f(1)}{h} = \left. \frac{d}{dx} x^2 \right|_{x=1} = 2x|_{x=1} = 2$$

The right-hand derivative of $f(x)$ at $x=1$ is

$$\lim_{h \rightarrow 0^+} \frac{f(1+h)-f(1)}{h} = \left. \frac{d}{dx} [k(x-1)+1] \right|_{x=1} = k$$

$f(x)$ is differentiable at $x=1$ if and only if the left-hand derivative at $x=1$ equals the right-hand derivative at $x=1$, i.e. when $k=2$.