§5.3.

29. (a) Orthonormal

(b) Neither

(c). Neither

35. $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ with $b_{11} + 3b_{21} + 2b_{12} + 4b_{22} = 0$

3. (a) If $A = [a_{ij}]$ then $(A, A) = \mathrm{Tr}(A^T A) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2 \geq 0$. Also $(A, A) = 0$ if and only if $a_{ij} = 0$, that is, if and only if $A = O$.

(b) If $B = [b_{ij}]$ then $(A, B) = \mathrm{Tr}(B^T A)$ and $(B, A) = \mathrm{Tr}(A^T B)$. Now

$$\mathrm{Tr}(B^T A) = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}^T a_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki} a_{ki},$$

and

$$\mathrm{Tr}(A^T B) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^T b_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki} b_{ki},$$

so $(A, B) = (B, A)$.

(c) If $C = [c_{ij}]$, then $(A + B, C) = \mathrm{Tr}[C^T(A + B)] = \mathrm{Tr}[C^T A + C^T B] = \mathrm{Tr}(C^T A) + \mathrm{Tr}(C^T B) = (A, C) + (B, C)$.

(d) $(cA, B) = \mathrm{Tr}(B^T(cA)) = c\,\mathrm{Tr}(B^T A) = c(A, B)$.

6. (a) $(p(t), q(t)) = \int_0^1 p(t)^2\, dt \geq 0$. Since $p(t)$ is continuous,

$$\int_0^1 p(t)^2\, dt = 0 \iff p(t) = 0.$$

(b) $(p(t), q(t)) = \int_0^1 p(t)q(t)\, dt = \int_0^1 q(t)p(t)\, dt = (q(t), p(t))$.

(c) $(p(t)+q(t), r(t)) = \int_0^1 (p(t)+q(t))r(t)\, dt = \int_0^1 p(t)r(t)\, dt + \int_0^1 q(t)r(t)\, dt = (p(t), r(t)) + (q(t), r(t))$.

(d) $(cp(t), q(t)) = \int_0^1 (cp(t))q(t)\, dt = c\int_0^1 p(t)q(t)\, dt = c(p(t), q(t))$.

7. (a) $\mathbf{0} + \mathbf{0} = \mathbf{0}$ so $(\mathbf{0}, \mathbf{0}) = (\mathbf{0}, \mathbf{0} + \mathbf{0}) = (\mathbf{0}, \mathbf{0}) + (\mathbf{0}, \mathbf{0})$, and then $(\mathbf{0}, \mathbf{0}) = 0$. Hence $\|\mathbf{0}\| = \sqrt{(\mathbf{0}, \mathbf{0})} = \sqrt{0} = 0$.

(b) $(\mathbf{u}, \mathbf{0}) = (\mathbf{u}, \mathbf{0} + \mathbf{0}) = (\mathbf{u}, \mathbf{0}) + (\mathbf{u}, \mathbf{0})$ so $(\mathbf{u}, \mathbf{0}) = 0$.

(c) If $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{v}$ in V, then $(\mathbf{u}, \mathbf{u}) = 0$ so $\mathbf{u} = \mathbf{0}$.

(d) If $(\mathbf{u}, \mathbf{w}) = (\mathbf{v}, \mathbf{w})$ for all $\mathbf{w}$ in V, then $(\mathbf{u} - \mathbf{v}, \mathbf{w}) = 0$ and so $\mathbf{u} = \mathbf{v}$.

(e) If $(\mathbf{w}, \mathbf{u}) = (\mathbf{w}, \mathbf{v})$ for all $\mathbf{w}$ in V, then $(\mathbf{w}, \mathbf{u} - \mathbf{v}) = 0$ or $(\mathbf{u} - \mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w}$ in V. Then $\mathbf{u} = \mathbf{v}$.

8. (a) 7. (b) 0. (c) -9.

10. (a) $\frac{13}{6}$. (b) 3. (c) 4.

12. (a) $\sqrt{22}$. (b) $\sqrt{18}$. (c) 1.

14. (a) $-\frac{1}{2}$. (b) 1. (c) $\frac{4}{7}\sqrt{3}$.

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19. $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}) = (\mathbf{u}, \mathbf{u})+2(\mathbf{u}, \mathbf{v})+(\mathbf{v}, \mathbf{v}) = \|\mathbf{u}\|^2+2(\mathbf{u}, \mathbf{v})+\|\mathbf{v}\|^2$. Thus $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2$ if and only if $(\mathbf{u}, \mathbf{v}) = 0$.

23. Let W be the set of all vectors in V orthogonal to $\mathbf{u}$. Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in W so that $(\mathbf{u}, \mathbf{v}) = 0$ and $(\mathbf{u}, \mathbf{w}) = 0$. Then $(\mathbf{u}, r\mathbf{v} + s\mathbf{w}) = r(\mathbf{u}, \mathbf{v}) + s(\mathbf{u}, \mathbf{w}) = r(0) + s(0) = 0$ for any scalars r and s.

39. If $\mathbf{u}$ and $\mathbf{v}$ are in R^n, let $\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Then

$$(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{n} a_i b_i = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

42. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal set, by Theorem 5.4 it is linearly independent. Hence, A is nonsingular. Since S is orthonormal,

$$(\mathbf{v}_i, \mathbf{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This can be written in terms of matrices as

$$\mathbf{v}_i \mathbf{v}_j^T = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

or as $AA^T = I_n$. Then $A^{-1} = A^T$. Examples of such matrices:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{3} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

43. Since some of the vectors $\mathbf{v}_j$ can be zero, A can be singular.

44. Suppose that A is nonsingular. Let $\mathbf{x}$ be a nonzero vector in R^n. Consider $\mathbf{x}^T(A^TA)\mathbf{x}$. We have $\mathbf{x}^T(A^TA)\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x})$. Let $\mathbf{y} = A\mathbf{x}$. Then we note that $\mathbf{x}^T(A^TA)\mathbf{x} = \mathbf{y}\mathbf{y}^T$ which is positive if $\mathbf{y} \neq \mathbf{0}$. If $\mathbf{y} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{0}$, and since A is nonsingular we must have $\mathbf{x} = \mathbf{0}$, a contradiction. Hence, $\mathbf{y} \neq \mathbf{0}$.

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10. $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}, \frac{\sqrt{2}}{2}\begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}$.

16. $\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{42}} & \frac{1}{\sqrt{42}} & \frac{2}{\sqrt{42}} & \frac{6}{\sqrt{42}}\end{bmatrix}$.

22. (a) $\sqrt{14}$. (b) $\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix}, \frac{1}{3}\begin{bmatrix}-2\\2\\-1\end{bmatrix} \right\}$. (c) $\left[\mathbf{v}\right]_T = \begin{bmatrix}\sqrt{5}\\3\end{bmatrix}$, so $\|\left[\mathbf{v}\right]_T\| = \sqrt{5+9} = \sqrt{14}$.

30. (a) $Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} 0.8944 & 0.4082 \\ -0.4472 & 0.8165 \\ 0 & 0.4082 \end{bmatrix}$, $R = \begin{bmatrix} \frac{5}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} 2.2361 & -2.2361 \\ 0 & 2.4495 \end{bmatrix}$.

(b) $Q = \begin{bmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} -0.5774 & 0 & 0.8165 \\ 0.5774 & -0.7071 & 0.4082 \\ 0.5774 & 0.7071 & 0.4082 \end{bmatrix}$.

$R = \begin{bmatrix} -\sqrt{3} & 0 & 0 \\ 0 & -\sqrt{8} & \sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix} \approx \begin{bmatrix} -1.7321 & 0 & 0 \\ 0 & -2.8284 & 1.4142 \\ 0 & 0 & -2.4495 \end{bmatrix}$.

(c) $Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & -\frac{5}{\sqrt{30}} \end{bmatrix} \approx \begin{bmatrix} 0.8944 & -0.4082 & -0.1826 \\ 0.4472 & 0.8165 & 0.3651 \\ 0 & 0.4082 & -0.9129 \end{bmatrix}$

$R = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{6} & -\frac{7}{\sqrt{6}} \\ 0 & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix} \approx \begin{bmatrix} 2.2361 & 0 & 0 \\ 0 & 2.4495 & -2.8577 \\ 0 & 0 & -0.9129 \end{bmatrix}$.

33. Let W be the subset of vectors in R^n that are orthogonal to $\mathbf{u}$. If $\mathbf{v}$ and $\mathbf{w}$ are in W then $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{w}) = 0$. It follows that $(\mathbf{u}, \mathbf{v}+\mathbf{w}) = (\mathbf{u}, \mathbf{v}) + (\mathbf{u}, \mathbf{w}) = 0$, and for any scalar c, $(\mathbf{u}, c\mathbf{v}) = c(\mathbf{u}, \mathbf{v}) = 0$, so $\mathbf{v}+\mathbf{w}$ and $c\mathbf{v}$ are in W. Hence, W is a subspace of R^n.