Solutions to HW 8

4.8

18)

(a) $[v]_T = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$, $[w]_T = \begin{bmatrix} 0 \\ 8 \\ -6 \end{bmatrix}$

(b) $P_{S \leftarrow T} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix}$

c) $[v]_S = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, $[w]_S = \begin{bmatrix} 8 \\ -4 \\ -2 \end{bmatrix}$

d) Same as (c)

e) $Q_{T \leftarrow S} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

f) Same as (a)

24) $S = \{v_1, v_2, v_3\}$ $T = \{w_1, w_2, w_3\}$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$P_{S \leftarrow T} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

$[v_1\ v_2\ v_3 \vdots w_1]$; $[v_1\ v_2\ v_3 \vdots w_2]$ and

$[v_1\ v_2\ v_3 \vdots w_3]$

Solving we get

$$T = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

41) a) From exercise 40 we have

$$M_S [v]_S = M_T [v]_T$$

From exercise 39 we know that M_S is nonsingular so

$$[v]_S = M_S^{-1} M_T [v]_T$$

Equation (3) is $[v]_S = P_{S \leftarrow T} [v]_T$,

so $\quad P_{S \leftarrow T} = M_S^{-1} M_T$

b) Since M_S and M_T are nonsingular, M_S^{-1} is nonsingular, so $P_{S \leftarrow T}$ as the product of 2 nonsingular matrices, is nonsingular

c) $M_S = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $M_T = \begin{bmatrix} 6 & 4 & 5 \\ 3 & -1 & 5 \\ 3 & 3 & 2 \end{bmatrix}$, $M_S^{-1} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ 1/3 & 1/3 & -2/3 \\ -2/3 & 1/3 & 4/3 \end{bmatrix}$

$$P_{S \leftarrow T} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

4.9.

6) (a)

$$\left\{(1,0,0,-33/7),\ (0,1,0,23/7),\ (0,0,1,-8/7)\right\}$$

(b)

$$\left\{(1,2,-1,3),\ (3,5,2,0),\ (0,1,2,1)\right\}$$

16) a)
$$\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consistent

b)
$$\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -13 \\ 3 \end{bmatrix}$$

Consistent.

43) Let rank $A = n$. Then Corollary 4.7 implies that A is nonsingular so $x = A^{-1}b$ is a solution. If x_1 and x_2 are solutions, then $Ax_1 = Ax_2$ and multiplying both sides by A^{-1}, we have $x_1 = x_2$. Thus $Ax = b$ has a unique solution.
Conversely, suppose that $Ax = b$ has a unique solution for every $n \times 1$ matrix b. Then the n linear systems $Ax = e_1$, $Ax = e_2, \dots$ $Ax = e_n$, where $e_1, e_2 \dots e_n$ are the columns of I_n, have solutions $x_1, x_2 \cdot x_n$. Let B be the matrix whose jth column is x_j. Then the n linear systems can

be written as $AB = I_n$. Hence, $B = A^{-1}$ so A is nonsingular and corollary 4.7 implies that rank $A = n$.

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