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4. (a) $\{[0 \ \ 0]\}$. (b) Yes. (c) No.

6. (a) A possible basis for $\ker L$ is $\{1\}$ and $\dim \ker L = 1$.
   (b) A possible basis for range L is $\{2t^3, t^2\}$ and $\dim \text{range}\, L = 2$.

8. (a) $\{-t^2 + t + 1\}$. (b) $\{t, 1\}$.

10. (a) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \right\}$. (b) $\left\{ \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

15. (a) $\dim(\ker L) = 1$, $\dim(\text{range } L) = 2$.

(b) $\dim(\ker L) = 1$, $\dim(\text{range } L) = 2$.

(c) $\dim(\ker L) = 0$, $\dim(\text{range } L) = 4$.

18. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. If L is invertible then L is one-to-one: from Theorem 6.7 it follows that $T = \{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$ is linearly independent. Since $\dim W = \dim V = n$, T is a basis for W. Conversely, let the image of a basis for V under L be a basis for W. Let $\mathbf{v} \neq \mathbf{0}_V$ be any vector in V. Then there exists a basis for V including $\mathbf{v}$ (Theorem 4.11). From the hypothesis we conclude that $L(\mathbf{v}) \neq \mathbf{0}_W$. Hence, $\ker L = \{\mathbf{0}_V\}$ and L is one-to-one. From Corollary 6.2 it follows that L is onto. Hence, L is invertible.

24. If L is one-to-one, then $\dim V = \dim \ker L + \dim \text{range}\, L = \dim \text{range}\, L$. Conversely, if $\dim \text{range}\, L = \dim V$, then $\dim \ker L = 0$.

29. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to $L(\mathbf{x}) = \mathbf{b}$. We show that $\mathbf{x}_1 - \mathbf{x}_2$ is in $\ker L$:

$$L(\mathbf{x}_1 - \mathbf{x}_2) = L(\mathbf{x}_1) - L(\mathbf{x}_2) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

31. From Theorem 6.6, we have $\dim \ker L + \dim \text{range}\, L = \dim V$.
   (a) If L is one-to-one, then $\ker L = \{\mathbf{0}\}$, so $\dim \ker L = 0$. Hence $\dim \text{range}\, L = \dim V = \dim W$ so L is onto.
   (b) If L is onto, then $\text{range}\, L = W$, so $\dim \text{range}\, L = \dim W = \dim V$. Hence $\dim \ker L = 0$ and L is one-to-one.

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2. (a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 2 & 0 & 2 \end{bmatrix}$.

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8. (a) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 4 & 3 & 0 & 3 \\ -6 & -5 & -4 & -3 \\ 3 & 3 & 7 & 0 \\ 8 & 6 & 4 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} 0 & 3 & 0 & 4 \\ -2 & -3 & -2 & -4 \\ 3 & 0 & 4 & 0 \\ 2 & 4 & 2 & 6 \end{bmatrix}$. (d) $\begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 3 & 7 & 0 \\ 4 & 3 & 0 & 3 \end{bmatrix}$.

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12. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be an ordered basis for U and $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_n\}$ an ordered ✓ basis for V (Theorem 4.11). Now $L(\mathbf{v}_j)$ for $j = 1, 2, \ldots, m$ is a vector in U, so $L(\mathbf{v}_j)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Thus $L(\mathbf{v}_j) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + \cdots + 0\mathbf{v}_n$. Hence,

$$\left[L(\mathbf{v}_j)\right]_T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(13) (a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$

20. (a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. (d) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

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23. Let $I\colon V \to V$ be the identity operator defined by $I(\mathbf{v}) = \mathbf{v}$ for $\mathbf{v}$ in V. The matrix A of I with respect ✓ to S and T is obtained as follows. The jth column of A is $\left[I(\mathbf{v}_j)\right]_T = \left[\mathbf{v}_j\right]_T$, so as defined in Section 3.7, A is the transition matrix $P_{T\leftarrow S}$ from the S-basis to the T-basis.

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9. $\begin{pmatrix} \frac{13}{2} & \frac{1}{2} & -\frac{3}{2} \\ -\frac{5}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

6. If $B = P^{-1}AP$, then $B^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$. Thus, A^2 and B^2 are similar, etc.

7. If $B = P^{-1}AP$, then $B^T = P^TA^T(P^{-1})^T$. Let $Q = (P^{-1})^T$, so $B^T = Q^{-1}A^TQ$.

8. If $B = P^{-1}AP$, then $\mathrm{Tr}(B) = \mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}(APP^{-1}) = \mathrm{Tr}(AI_n) = \mathrm{Tr}(A)$.

10. Possible answer: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

11. (a) If $B = P^{-1}AP$ and A is nonsingular then B is nonsingular.
(b) If $B = P^{-1}AP$ then $B^{-1} = P^{-1}A^{-1}P$.

12. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

14. $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$B = Q^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

16. A and O are similar if and only if $A = P^{-1}OP = O$ for a nonsingular matrix P.

17. Let $B = P^{-1}AP$. Then $\det(B) = \det(P^{-1}AP) = \det(P)^{-1}\det(A)\det(P) = \det(A)$.