Some Linear Algebra Notes

An mxn **linear system** is a system of m linear equations in n unknowns x_i, $i = 1, \ldots, n$:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \quad \cdots + a_{1n}x_n \quad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \quad \cdots + a_{2n}x_n \quad &= b_2 \\
\ddots \qquad\qquad &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \quad \cdots + a_{mn}x_n \quad &= b_m
\end{aligned}$$

The **coefficients** a_{ij} give rise to the rectangular matrix $A = (a_{ij})_{mxn}$ (the first subscript is the row, the second is the column). This is a matrix with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \ddots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

A **solution** to the linear system is a sequence of numbers $s_1, s_2, \cdots, s_n$, which has the property that each equation is satisfied when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$.

If the linear system has a nonzero solution it is **consistent**, otherwise it is **inconsistent**.

If the right hand side of the linear system constant 0, then it is called a **homogeneous** linear system. The homogeneous linear system always has the **trivial solution** $x = 0$.

Two linear systems are **equivalent**, if they both have exactly the same solutions.

Def 1.1: An mxn **matrix** A is a rectangular array of mn real or complex numbers arranged in m (horizontal) rows and n (vertical) columns.

Def 1.2: Two mxn matrices $A = (a_{ij})$ and $B = (b_{ij})$ are **equal**, if they agree entry by entry.

Def 1.3: The mxn matrices A and B are added entry by entry.

Def 1.4: If $A = (a_{ij})$ and r is a real number, then the **scalar** multiple of r and A is the matrix $rA = (ra_{ij})$.

If $A_1, A_2, ..., A_k$ are mxn matrices and $c_1, c_2, ..., c_k$ are real numbers, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k$$

is a **linear combination** of the A's with coefficients $c_1, c_2, ..., c_k$.

Def 1.5: The **transpose** of the mxn matrix $A = (a_{ij})$ is the nxm matrix $A^T = (a_{ji})$.

Def 1.6: The **dot product** or **inner product** of the n-**vectors** $a = (a_i)$ and $b = (b_i)$ is

$$a \cdot b = a_1b_1 + a_2b_2 + ... + a_nb_n$$

.
Example: Determine the values of x and y so that $v \cdot w = 0$ and $v \cdot u = 0$,
where $v = \begin{bmatrix} x \\ 1 \\ y \end{bmatrix}$, $w = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, and $u = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$.

Def 1.7: If $A = (a_{ij})$ is an m x p matrix and $B = (b_{ij})$ a p x n matrix they can be multiplied and the ij entry of the m x n matrix $C = AB$:

$$c_{ij} = (a_{i*})^T \cdot (b_{*j})$$

. Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 2 & -1 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}.$$

If possible, find AB, BA, A^2, B^2.
Which matrix rows/columns do you have to multiply in order to get the $3,1$ entry of the matrix AB?
Describe the first row of AB as the product of rows/columns of A and B.

The linear system (see beginning) can thus be written in matrix form $Ax = b$.

Write it out in detail.

A is called the **coefficient matrix** of the linear system and the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ & \ddots & & \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_n \end{bmatrix}.$$

is called the **augmented matrix** of the linear system.

Note: $Ax = b$ is consistent if and only if b can be expressed as a linear combination of the columns of A with coefficients x_i.

Theorem 1.1 Let A, B, and C be mxn matrices, then
(a) $A + B = B + A$
(b) $A + (B + C) = (A + B) + C$
(c) there is a unique mxn matrix O such that for any mxn matrix A: $A + O = A$
(d) for each mxn matrix A, there is aunique mxn matrix D such that
$A + D = O$
$D = -A$ is the negative of A.

Theorem 1.2 Let A, B, and C be matrices of the appropriate sizes, then
(a) $A(BC) = (AB)C$

(b) $(A+B)C = AC + BC$
(c) $C(A+B) = CA + CB$

Prove part (b)

Theorem 1.3 Let r, s be real numbers and A, B matrices of the appropriate sizes, then
(a) $r(sA) = (rs)A$
(b) $(r+s)A = rA + sA$
(c) $r(A+B) = rA + rB$
(d) $A(rB) = r(AB) = (rA)B$

Theorem 1.4 Let r be a scalar, A, B matrices of appropriate sizes, then
(a) $(A^T)^T = A$
(b) $(A+B)^T = A^T + B^T$
(c) $(AB)^T = B^T A^T$
(d) $(rA)^T = rA^T$

prove part (c)

Note:
(a) AB need not equal BA.
(b) AB may be the zero matrix O with A not equal O and B not equal O.
(c) AB may equal AC with B not equal C.

Find two different 2 x 2 matrices A such that $A^2 = 0$.

Find three different 2 x 2 matrices A, B and C such that $AB = AC$, $A \neq 0$ and $B \neq C$.

Def 1.8:
indent A matrix $A = [a_{ij}]$ is a **diagonal matrix** if $a_{ij} = 0$ for $i \neq j$.
A **scalar matrix** is a diagonal matrix whose diagonal entries are equal.
The scalar matrix $I_n = d_{ij}$, where $d_{ii} = 1$ and $d_{ij} = 0$ for $i \neq j$ is called the nxn **identity matrix**.

Example: If square matrices A and B satisfy that $AB = BA$, then $(AB)^p = A^p B^p$.

Def 1.9:
A matrix A with real enties is **symmetric** if $A^T = A$.
A matrix with real entries is **skewsymmetric** if $A^T = -A$.

Let $B = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}$
compute BB^T and $B^T B$. What can you say about them?

An nxn matrix A is **upper triangular** if $a_{ij} = 0$ for $i > j$, **lower triangular** if $a = 0$ for $i < j$.
Given an mxn matrix $A = [a_{ij}]$. If we cross out some, but not all of it's rows and columns, we get a

submatrix of A.
A matrix can be **partitioned** into submatrices by drawing horizontal lines between rows and vertical lines between columns.

Def 1.10: An nxn matrix A is **nonsingular** or **invertible**, if there exists an nxn matrix B such that

$AB = BA = I_n$

B would then be the inverse of A

Otherwise A is **singular** or **noninvertible**.

Remark: At this point, we have not shown that if $AB = I_n$, then $BA = I_n$, this will be done in Theorem 2.11. In the mean time we assume it.

If $D = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find D^{-1}.

Theorem 1.5 The inverse of a matrix, if it exists is unique.
Prove it.

If A is a nonsingular matrix whose inverse is $\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, find A.

Theorem 1.6 If A and B are both nonsingular nxn matrices then AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Prove it.

Corollary 1.1 If $A_1, A_2, ..., A_r$ are nonsingular nxn matrices, then $A_1A_2...A_r$ is nonsingular and

$$(A_1A_2 \cdots A_r)^{-1} = A_r^{-1} \cdots A_2^{-1}A_1^{-1}$$

. **Theorem 1.7** If A is a nonsingular matrix, then A^{-1} is nonsingular and

$$(A^{-1})^{-1} = A.$$

why?

Theorem 1.8 If A is a nonsingular matrix, then A^T is nonsingular and

$$(A^{-1})^T = (A^T)^{-1}.$$

Show that if the matrix A is symmetric and nonsingular, then A^{-1} is symmetric.

Note: If A is a nonsingular nxn matrix. Then

(a) the linear system $Ax = b$ has the unique solution $x = A^{-1}b$. Why?

(b) the homogeneous linear system Ax = 0 has the unique solution x = 0. Why?

Consider the linear system $Ax = b$, where $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$.

Find a solution if $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Suppose A an mxn matrix, x an n-vector, i.e in $\mathbb{R}^n$.
Then $Ax = y$ is an m-vector, y in $\mathbb{R}^m$. So A represents a **matrix transformation** f,

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$
$$x \longrightarrow y = Ax.$$

For u in $\mathbb{R}^n$: $f(u) = Au$ is the **image** of u.

$\{f(u) = Au | u \in \mathbb{R}^n\}$ is the **range** of f.

For the given matrix transformations f and vectors u, find $f(u)$.
Geometrically (draw pictures), what does f do?

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, u = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

(a) was a **reflection** on the x-axis.
(b) was **dilation** by a factor of 3. If the factor is < 1, it's called a **contraction**.
(c) was a rotation around the origin by angle θ.
(d) was a vertical **projection** onto the yz-plane.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$,
$x \longrightarrow y = Ax.$
Show that:
(a) $f(u + v) = f(u) + f(v)$, for $u, v \in \mathbb{R}^n$.
(b) $f(cu) = cf(u)$, where $c \in \mathbb{R}$, $u \in \mathbb{R}^n$.

Def 2.1 An mxn , matrix is said to be in **reduced row echelon form** if it satisfies the following properties:
(a) all zero rows, if there are any, are at the bottom of the matrix.
(b) the first nonzero entry from the left of a nonzero row is a 1.This entry is called a leading one of its row.
(c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.
(d) If a column contains a leading one, then all other entries in that column are zero.

An mxn , matrix is in **row echelon form**, if it satisfies properties (a), (b), and (c).
Similar definition for column echelon form.

What can you say about these matrices?

$$(1) \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$(2) \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$(3) \quad A = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Def 2.2 An **elementary row (column) operation** on a matrix A is one of these:
(a) interchange of two rows
(b) multiply a row by a nonzero number
(c) add a multiple of one row to another.

For the matrix $A = \begin{bmatrix} -1 & 1 & -1 & 0 & 3 \\ -3 & 4 & 1 & 1 & 10 \\ 4 & -6 & -4 & -2 & -14 \end{bmatrix}$. Find
(a) a row-echelon form
(b) the reduced row-echelon form

Def 2.3 An mxn , matrix B is **row (column) equivalent** to an mxn , matrix A, if B can be produced by applying a finite sequence of elementary row (column) operations to A.

Theorem 2.1 Every nonzero mxn matrix $A = [a_{ij}]$ is row (column) equivalent to a matrix in row (column) echelon form.

Theorem 2.2 Every nonzero mxn , matrix $A = [a_{ij}]$ is row (column) equivalent to a unique matrix in reduced (column) row echelon form.
The uniqueness proof is involved, see Hoffman and Kunze, Linear Algebra, 2nd ed.

Note: the row echelon form of a matrix is not unique.
Why?

Theorem 2.3 Let $Ax = b$ and $Cx = d$ be two linear systems, each of m equations in n unknowns. If the augmented matrices $[A|b]$ and $[C|d]$ are row equivalent, then the linear systems are equivalent, i.e. they have exactly the same solutions.

Corollary 2.1 If A and C are row equivalent mxn matrices, then the homogeneous systems $Ax = 0$ and $Cx = 0$ are equivalent.

Find the solutions (if they exist) for these augmented matrices:

(a) $$A = \begin{bmatrix} 1 & 1 & -1 & 0 & \vdots & 3 \\ 0 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 1 & \vdots & -1 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 1 & 1 & -1 & 0 & \vdots & 3 \\ 0 & 1 & 0 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 0 & \vdots & -1 \end{bmatrix}$$

(c) $$A = \begin{bmatrix} 1 & 1 & -2 & 0 & \vdots & 3 \\ 0 & 0 & 0 & 1 & \vdots & -1 \end{bmatrix}$$

Theorem 2.4 A homogeneous system of m linear equations in n unknowns always has a nontrivial solution if $m < n$, that is, if the number of unknowns exceeds the number of equations.

Given $Ax = 0$ with $A = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{bmatrix}$
Find the solution set for $Ax = 0$.

Gaussian elimination: transform $[A|b]$ to $[C|d]$, where $[C|d]$ is in row echelon form.

Gauss Jordan reduction: transform $[A|b]$ to $[C|d]$, where $[C|d]$ is in reduced row echelon form.

Find an equation relating a, b and c, so that the linear system:

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 3y + 3z &= b \\ 5x + 9y - 6z &= c \end{aligned}$$

is consistent for any values of a, b and c, that satisfy that equation.

Let $Ax = b$, $b \neq 0$, be a consistent linear system.
Show that if x_p is a particular solution to the given nonhomogeneous system and x_h is a solution to the associated homogeneous system $Ax = 0$, then $x_p + x_h$ is a solution to the given system $Ax = b$.

Ethane is a gas similar to methane that burns in oxygen to give carbon dioxide gas and steam. The steam condenses to form water droplets. The chemical equation for this reaction is:

$$C_2H_6 + O_2 \longrightarrow CO_2 + H_2O$$

Balance this equation.

Def 2.4 an **elementary matrix** is a matrix obtained from the identity matrix by performing a single elementary row operation.

Find the matrix E otained from the identity matrix I_3 by the row manipulation (3)-2(1) $\rightarrow$ (3) (i.e. the third row is replaced by row 3 - 2 * row 1).

For the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$,

(a) left multiply A with $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

(b) left multiply the matrix you got from part (a) with $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

(c) left multiply the matrix you got from part (b) with $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

(d) left multiply the matrix you got from part (c) with $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$,

give a further sequence of elementary matrices that will transform A to
- row-echelon form,
- reduced row echelon form

Theorem 2.5 Perform an elementary row operation (with matrix E) on mxn , matrix A to yield matrix B. Then $B = EA$.

Theorem 2.6 Let A, B be mxn , matrices. Equivalent:
(a) A is row equivalent to B.
(b) There exist elementary matrices $E_1, E_2, ..., E_k$, such that

$$B = E_k E_{k-1} ... E_1 A.$$

Theorem 2.7 An elementary matrix E is nonsingular, and its inverse is an elementary matrix of the same type.

Lemma 2.1 Let A be an nxn matrix and suppose the homogeneous system $Ax = 0$ has only the trivial solution $x = 0$. Then A is row equivalent to I_n.

Theorem 2.8 A is nonsingular if and only if A is the product of elementary matrices.

Corollary 2.2 A is nonsingular if and only if A is row equivalent to I_n.

Theorem 2.9 Equivalent:
(a) The homogeneous system of n linear equations in n unknowns $Ax = 0$ has a nontrivial solution.
(b) A is singular.

Theorem 2.10 Equivalent:

(a) nxn matrix A is singular.
(b) A is row equivalent to a matrix that has a row of zeroes.

Theorem 2.11 Let A, B be nxn matrices such that $AB = I_n$, then $BA = I_n$ and $B = A^{-1}$.

Prove: If A and B are nxn matrices and AB nonsingular, then A and B are each also nonsingular.

Def 2.5 A, B mxn , matrices. A is **equivalent** to B, if we can obtain B from A by a finite sequence of elementary row and column operations.

Theorem 2.12 If A is a nonzero mxn , matrix, then A is equivalent to a partitioned matrix of the form:

$$\begin{bmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix}$$

Theorem 2.13 Let A, B be mxn , matrices. Equivalent:
(a) A is equivalent to B.
(b) $B = PAQ$ (P = product of elementary row matrices, Q = product of elementary column matrices).

Theorem 2.14 Let A be an nxn matrix. Equivalent:
(a) A is nonsingular.
(b) A is equivalent to I_n .

Def 3.1 Let $S = 1, 2, ..., n$ in this order. A **rearrangement** $j_1 j_2 ... j_n$ of the elements of S is a permutation of S.

How many permutations of the set $S = 1, 2, 3$ are there?
How many permutations of the set $S = 1, 2, ..., n$ are there?

A permutation $j_1 j_2 ... j_n$ is said to have an **inversion** if a larger integer j_r precedes a smaller one, j_s.
A permutation is even if the total number of inversions is **even**, or **odd** if the total number of inversions in it is odd.

Is the permutation 43512 even or odd?

Def 3.2 Let $A = [a_{ij}]$ be an nxn matrix. The **determinant** function **det** is defined by

$$det(A) = \sum (\pm) a_{1j_1} a_{2j_2} ... a_{nj_n}$$

, where the summation is over all permutations $j_1 j_2 ... j_n$ of the set S.
The sign is taken as + or - according to whether the permutation $j_1 j_2 ... j_n$ is even or odd.

Compute the determinant of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

by writing out the permutations of $S = 1, 2, 3$ and their parity.

Compute

$$\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}.$$

Theorem 3.1 If A is a matrix, then $\det(A) = \det(A^T)$.

Theorem 3.2 If matrix B results from matrix A by interchanging two different rows(columns) of A, then $\det(B)$ = - $\det(A)$.

Proof idea: The number of inversions in $j_1 j_2 ... j_r ... j_s ... j_n$ differs by an odd number from the number of inversions in $j_1 j_2 ... j_s ... j_r ... j_n$. Thus the signs of the terms in B are the negatives of the signs in the terms of A.

Find the number inversions in 43512 and in 41532 (switch of positions 2 and 4).

Theorem 3.3 If two rows (columns) of A are equal, then $\det(A) = 0$.

neat proof.

Theorem 3.4 If a row (column) of A consists entirely of zeros, then $\det(A) = 0$.

Theorem 3.5 If B is obtained from A by multiplying a row (column) of A by a real number k, then $\det(B)$ = k $\det(A)$.

Proof idea: in each summand of the determinant, there is a factor of k, coming from the a_{i*}, where i was the row that was multiplied by k.

Theorem 3.6 If $B = [b_{ij}]$ is obtained from $A = [a_{ij}]$ by adding to each element of the sth (column) of A, k times the corresponding element of the rth row(column), $r \neq s$, of A, then $\det(B) = \det(A)$.
Proof: see text.

So now we know that elementary row operations change the determinant of A in predictable ways.

Compute the determinant of $\begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 4$ find $\begin{vmatrix} a_1 & a_2 & a_3 - a_2 \\ b_1 & b_2 & b_3 - b_2 \\ \frac{1}{2}c_1 & \frac{1}{2}c_2 & \frac{1}{2}c_3 - \frac{1}{2}c_2 \end{vmatrix}$

Theorem 3.7 If a matrix $A = [a_{ij}]$ is upper(lower) triangular, then
$\det(A) = a_{11}a_{22}...a_{nn}$;
that is the determinant of a triangular matrix is the product of the elements on the main diagonal.

Proof idea: The only nonzero term is the one for the permutation $123...n$.

Notice how we described the effect of doing an elementary row operation on A on the determinant. Combine this into a crucial Lemma.

Lemma 3.1 If E is an elementary matrix, then
$\det(EA) = det(E)det(\mathrm{A})$, and
$\det(AE) = det(\mathrm{A})det(E)$.

Proof:
(i) for a row switch: Theorem 3.2.
(ii) for multiplying row i by a constant: Theorem 3.5.
(iii) for a multiple of row j to row i: Theorem 3.6.

Theorem 3.8 If A is an nxn matrix, equivalent
(a) A is nonsingular.
(b) $\det(A) \neq 0$.

Is this matrix nonsingular?

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

Proof: follows from Theorem 2.8 (A is a product of elementary matrices) and Lemma 3.1.

Corollary 3.1 A an nxn matrix. Equivalent:

(a) $Ax = 0$ has a nontrivial solution.
(b) $\det(A) = 0$.

Proof idea: A is row equivalent to a matrix with at least one row of zeros. This means that the equation system has the same solution set as (is equivalent to) one with more variables than equations.

Theorem 3.9 If A, B are nxn matrices, then
$\det(AB) = \det(A)\det(B)$.

Corollary 3.2 If A is nonsingular, then $\det(A^{-1}) = \frac{1}{det(A)}$.

Proof: $A * A^{-1} = I_n$.
Take the determinant on both sides.
The determinant of I_n is 1.
The determinant of A is nonzero by Theorem 3.8.
$det(A * A^{-1}) = det(A) * det(A^{-1})$ by Theorem 3.9.
Hence $\det(A^{-1}) = \frac{1}{det(A)}$.

Def 6.6 Matrices A, B are **similar**, if there is a nonsingular matrix P, such that $B = P^{-1}AP$.

Corollary 3.3 If A, B are similar matrices, then $\det(A) = \det(B)$.
Proof: you try.

Is $|A + B| = |A| + |B|$? (example 11 in text)

Show: if A is nxn skew symmetric and n odd, then $\det A = 0$.

Use Theorem 3.8 to determin all values of t so that this following matrix is nonsingular:

$$\begin{vmatrix} t & 0 & 0 & 1 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 1 & 0 & 0 & t \end{vmatrix}.$$

Show that if A is an nxn skew symmetric matrix, n odd, then $\det(A) = 0$.

If A is a nonsingular matrix such that $A^2 = A$, what is $\det(A)$?

Let A be a 3x3 matrix with $\det(A)$=3.
  (a) Waht is the reduced row echelon form to which A is row equivalent.
  (b) How many solutionsdoes the homogeneous system $Ax = 0$ have?

Def 3.3 Let $A = [a_{ij}]$ be an nxn matrix. Let M_{ij} be the $(n-1)$x$(n-1)$ submatrix of A obtained by deleting the ith row and the jth column of A. The determinant $\det(M_{ij})$ is called the **minor** of a_{ij}.

Compute the minors of $A = \begin{bmatrix} -1 & 2 & 3 \\ -2 & 5 & 4 \\ 0 & 1 & -3 \end{bmatrix}$.

Def 3.4 Let $A = [a_{ij}]$ be an nxn matrix. The **cofactor** A_{ij} of a_{ij} is defined as $A_{ij} = (-1^{i+j}) \det(M_{ij})$.

Compute the cofactors for the above matrix A.

Theorem 3.10 Let $A = [a_{ij}]$ be an nxn matrix. Then
  $\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in}$
    (expansion along ith row)

  $\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + ... + a_{nj}A_{nj}$
    (expansion along jth column)

Prove it for A a 3x3 matrix expanded along the 1st row.

Compute the determinant of the above matrix by expanding along the 3rd row, and then by expanding along the 1st column.

Find all values of t for which $\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t+2 & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0$.

Application: The area of the triangle (x_1, y_1), (x_2, y_2), $(x_3, y_3) = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Theorem 3.11 Let $A = [a_{ij}]$ be an nxn matrix, then

$$\begin{aligned} &a_{i1}A_{k1} + a_{i2}A_{k2} + ... + a_{in}A_{kn} = 0 \text{ for } i \neq k \\ &a_{1j}A_{1k} + a_{2j}A_{2k} + ... + a_{nj}A_{nk} = 0 \text{ for } j \neq k \end{aligned}$$

Def 3.5 Let $A = [a_{ij}]$ be an nxn matrix. The nxn matrix $adjA$, called the **adjoint of** A, is the matrix whose (i,j)th entry is the cofactor A_{ji} of a_{ji}.

Theorem 3.12 Let $A = [a_{ij}]$ be an nxn matrix, then $A(adjA) = (adjA)A = \det(A)I_n$.

Corollary 3.4 Let A be an nxn matrix and $\det(A) \neq 0$,
then $A^{-1} = \frac{1}{detA} * (adjA)$.

Theorem 3.13 (Cramer's Rule) Let A be an nxn matrix and $Ax = b$ and det $A \neq 0$, then the system has the unique solution

$$x_1 = \frac{detA_1}{detA}, x_2 = \frac{detA_2}{detA}, ..., x_n = \frac{detA_1}{detA}$$

where A_i is the matrix obtained from A by replacing the ith column of A by b.

Def 4.1 A **vector** x **in the plane** is a 2x1 matrix $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1, x_2 are real numbers, called the **components, (entries)** of x.

Similarly for vectors in 3-space: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $x_i \in \mathbb{R}$, i = 1,2,3.

Determine the components pf the vector PQ, where $P = (-2, 2, 3)$, $Q = (-3, 5, 2)$.

Def 4.2 Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, be two vectors in the plane. The **sum** of the vectors u and v is the vector $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$.

Def 4.3 If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, and c is a scalar (a real number), then the **scalar multiple** cu of u by c is the vector $\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$.

Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $y = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $z = \begin{bmatrix} r \\ 4 \end{bmatrix}$, and $u = \begin{bmatrix} -2 \\ s \end{bmatrix}$. Find r, s so that

(a) $z = 2x$
(b) $\frac{3}{2}u = y$
(c) $z + u = x$

Def 4.4 A **real vector space** is a set V of elements on which we have two operations $\oplus$ and $\odot$ defined with these properties:

(a) if u, v are elements in V , then $u \oplus v$ is in V (closed under $\oplus$):
(i) $u \oplus v = v \oplus u$ for all $u, v \in V$
(ii) $u \oplus (v \oplus w) = (u \oplus v) \oplus w$ for $u, v, w \in V$
(iii) there exists an element $0 \in V$ such that $u \oplus 0 = 0 \oplus u = 0$ for $u \in V$.
(iv) for each $u$in V there exists an element $-u \in V$ such that $u \oplus (-u) = (-u) \oplus u = 0$.

(b) If u is any element in V and c is a real number, then $c \odot u$ (or cu) is in V (V is closed under scalar multiplication).
(i) $c \odot (u \oplus v) = c \odot u \oplus c \odot v$ for any $u, v \in V$, c a real number
(ii) $(c + d) \odot u = c \odot u \oplus d \odot u$ for any $u \in V$, c, d real numbers
(iii) $c \odot (d \odot u) = (cd) \odot u$ for any $u \in V$, c, d real numbers
(iv) $1 \odot u = u$ for any $u \in V$

Problem: Let V be the set of all polynomials of exactly degree 2. Is V closed under addition and scalar multiplication?

Problem: V = set of all 2x2 matrices $A = \begin{bmatrix} a & b \\ 2b & d \end{bmatrix}$, under $\oplus$ = standard addition, and $\odot$ = scalar multiplication.

(a) Is V closed under addition?
(b) Is V closed under scalar multiplication?
(c) Is there a 0-vectore in V?
(d) If A is in V, is there a $-A$ in V?
(e) Is V a vector space?

Theorem 4.2 If V is a vector space, then

(a) $0 \odot u = u$ for any $u \in V$
(b) $c \odot 0 = 0$ for any scalar c
(c) if $c \odot u = 0$, then either $c = 0$ or $u = 0$.
(d) $(-1) \odot u = -u$ for any vector $u \in V$

Problem: Let V be the set of all ordered triples of real numbers with the operations
$(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$
$r \odot (x, y, z) = (x, 1, z)$
Is V a vector space?

Problem: Is this a vector space: the set of all 2x1 matrices $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $u \leq 0$ with the usual operations in $\mathbb{R}^2$?

Problem: $\mathbb{R}^n$ with $u \oplus v = u + v$ and $c \odot u = cu$ is a vector space.

Problem: $\mathbb{R}$ with $u \oplus v = u + v$ and $c \odot u = cu$ is a vector space.

Problem: The set of all polynomials of degree $\leq n$ with $p \oplus q = p + q$ and $c \odot p = cp$ is a vector space.

Problem: Consider the differential equation $y" - y' + 2y = 0$. A solution is a real valued function f satisfying the equation. Let V be the set of all solutions to the given differential equation. Define $(f \oplus g)(t) = f(t) + g(t)$ and $(c \odot f)(t) = cf(t)$. Then V is a vector space.

Show: there is exactly one zero vector in a vector space.

Def 4.5 Let V be a vector space and W a nonempty subset of V.
If W is a vector space with respect to the operations in V, then W is called a **subspace** of V.

Theorem 4.3 Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold:

(a) if u, v are in W, then $u + v$ is in W
(b) if c is a real number and u is any vector in W, then cu is in W.

Problem: Is this a subspace?
(a) $\{(x, y, 0)|x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$

(b) $\{(x, y, 1)|x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$

(c) $\{(x, y)|x, y \in \mathbb{R}, x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$

(d) $\{(x, x, y)|x, y \in \mathbb{R}\}$ in $\mathbb{R}^3$

(e) $\{(x, y, z, w)|x, y, z, w \in \mathbb{R}, x - y = 2\}$ in $\mathbb{R}^4$

(f) $\{\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix} |a, b, c, d \in \mathbb{R}, a - b = 0\}$ in $M_{2,3}$

(g) $\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} |a, b, c, d \in \mathbb{R}, a + b + c + d = 0\}$ in $M_{2,2}$

(h) $\{f \in C(-\infty, \infty)|f(0) = 5\}$ in $C(-\infty, \infty)$.

(i) $\{f \in C(-\infty, \infty)|f$ bounded on $[a, b]\}$ in $C(-\infty, \infty)$.

The **nullspace** of a matrix A are all the vectors v, such that $Av = 0$.

Question: If A is a singular matrix, what can you say about the nullspace of A?

Definition 4.6 Let $v_1, v_2, ..., v_n$ be vectors in vector space V. A vector v is a **linear combination** of $v_1, v_2, ..., v_n$, if

$$v = a_1 v_1 + a_2 v_2 + ... + a_n v_n$$

Problem: The set W of all 2x2 symmetric matrices is a subspace of M_{22}. Find three 2x2 matrices v_1, v_2, v_3, so that every vector in W can be expressed as a linear combination of v_1, v_2, v_3.

Definition 4.7 If $S = \{v_1, v_2, ..., v_n\}$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by **span** S or span $\{v_1, v_2, ..., v_n\}$.

Problem: Why is $S = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$ not a spanning set for M_{22}?

Theorem 4.4 Let $S = \{v_1, v_2, ..., v_k\}$ be a set of vectors in a vector space V. Then span S is a subspace of V.
What do you have to show in order to prove this?

Problem: Does $p(t) = -t^2 + t - 4$ belong to the span $\{t^2 + 2t + 1, t^2 + 3, t - 1\}$?

Definition 4.8 Let S be a set of vectors in a vector space V.
If every vector in V is a linear combination of the vectors in S, then S is said to **span** V, or V is **spanned** by the set S; that is, span $S = V$.

Problem: Do these vectors span $\mathbb{R}^4$?
$\{[1100], [12-11], [0011], [2121]\}$

Problem: Does this set of polynomial span $\mathbb{P}_2$?
$\{t^2 + 1, t^2 + t, t + 1\}$

Problem: Find the set of all vectors spanning the nullspace of A: $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}$.

Problem: The set W of all 3x3 matrices A with trace 0 is a subspace of M_{33}. Determine the subset S of W, so that $W = \text{span} S$.

Definition 4.9 The vectors $v_1, v_2, ..., v_n$ in vector space V are said to be **linearly dependent**, if there exist constants $a_1, a_2, ..., a_n$, not all zero such that

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0.$$

Otherwise $v_1, v_2, ..., v_n$ are **linearly independent** if, whenever $a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0$, then

$$a_1 = a_2 = ... = a_n = 0.$$

Question: Is $S = \{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix} \}$ linearly independent in $\mathbb{R}^4$?

Theorem 4.5 Let $S = \{v_1, v_2, ..., v_n\}$ be a set of n vectors in $\mathbb{R}^n$.
Let A be a matrix whose columns (rows) are elements of S.
Then S is linearly independent if and only if $\det(A) \neq 0$.

Problem: The augmented matrix is derived from equation (1). Is S linearly independent?

$$\begin{bmatrix} 2 & 1 & 3 & 2 & | & 0 \\ -1 & 0 & 0 & 1 & | & 0 \\ 1 & -1 & 2 & 1 & | & 0 \\ 5 & 1 & 8 & 5 & | & 0 \end{bmatrix}$$

Problem: Given $A|0$, where A is 5x5 and nonsingular. Is S linearly independent?

Theorem 4.6 Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2 .
Then the following are true:
(a) If S_1 is linearly dependent, so is S_2.
(b) If S_2 is linearly independent, so is S_1 .

So: subsets of linearly independent sets are linearly independent, and
supersets of linearly dependent sets are linearly dependent.

Theorem 4.7 The nonzero vectors $v_1, v_2, ..., v_n$ in a vector space V are linearly dependent if and only if one of the vectors v_j $(j \geq 2)$ is a linear combination of the preceding vectors $v_1, v_2, ..., v_{j-1}$.

Problem: Are these vectors linearly independent? If not, express one of the vectors as a linear combination of the others.
(a) $[110], [023], [123], [366]$
(b) t^2, t, e^t
(c) $\cos^2 t, \sin^2 t, \cos 2t$

Problem: Suppose $S = \{v_1, v_2, v_3\}$ is a linearly independent set of vectors in a vector space V. Prove that $T = \{w_1, w_2, w_3\}$ is also linearly independent when $w_1 = v_1 + v_2 + v_3, w_2 = v_2 + v_3, w_3 = v_3$.

Problem: Suppose $\{v_1, v_2, ..., v_n\}$ is a linearly independentset of vectors in $\mathbb{R}^n$. Show that if A is an nxn nonsingular matrix, then $\{Av_1, Av_2, ..., Av_n\}$ is linearly independent.

Definition 4.10 The vectors in a vector space V are said to form a **basis** for V if

(a) $v_1, v_2, ..., v_k$ span V and
(b) $v_1, v_2, ..., v_k$ are linearly independent.

The **Natural (standard) basis** in $\mathbb{R}^n$: $\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \}$.

Is $S = \{-t^2 + t + 2, 2t^2 + 2t + 3, 4t^2 - 1\}$ a basis for $\mathbb{P}_2$?

Theorem 4.8 If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for the vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.

If $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$ form a basis for $\mathbb{R}^3$, express $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ in terms of them.

Theorem 4.9 Let $S = \{v_1, v_2, \dots, v_n\}$ be a set of nonzero vectorsin a vector space V and let $W =$ span S. Then some subset of S is a basis for W.

Let W be the subspace of $\mathbb{P}_3$ spanned by $S = \{t^3 + t^2 - 2t + 1, t^2 + 1, t^3 - 2t, 2t^3 + 3t^2 - 4t + 3\}$ Find a basis for W.

Theorem 4.10 If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for vector space V and $T = w_1, w_2, \dots, w_r$ is a linearly independent set of vectors in V, then $r \leq n$.

Problem: Find a basis for the subspace of $\mathbb{R}^3$ given by all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $2a + b - c = 0$.

Corrollary 4.1 If $S = \{v_1, v_2, \dots, v_n\}$ and $T = \{w_1, w_2, \dots, w_m\}$ are bases for a vector space V, then $n = m$.

Definition 4.11 The **dimension** of a nonzero vector space V (dim V) is the number of vectors in a basis for V.
The dimension of the trivial vector space 0 is 0.

Find the dimension of the subspace of $\mathbb{R}_4$ spanned by $\S = \{[1001], [0100], [1111], [0111]\}$.

Definition 4.12 Let S be a set of vectors in a vector space V.
A subset T of S is called a **maximal independent** subset of S, if T is a linearly independent set of vectors that is not properly contained in any other linearly independent subset of S.

Corrollary 4.2 If the vector space V has dimension n, then a maximal independent subset of vectors in contains n vectors.

Problem: Prove that the vector space $\mathbb{P}$ is not finite dimensional.

Problem: Let V be a n-dimsnsional vector space.. Show that any $n + 1$ vectors in V form a linearly dependant set.
Corrollary 4.3 If a vector space V has dimension n, then a minimal spanning set (if it does not properly contain any other set spanning V) for V contains n vectors.

Corollary 4.4 If a vector space V has dimension n, then any subset of $m > n$ vectors must be linearly dependent.

Corollary 4.5 If a vector space V has dimension n, then any subset of $m < n$ vectors cannot span V. Prove it.

Problem: Give an example of a 2-dimensional subspace of $\mathbb{P}_3$.

Theorem 4.11 If S is a linearly independent set of vectors in a finite dimensional vector space V, then there is a basis for V that contains S.

Problem: Find a basis for $\mathbb{R}^3$ that includes the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

Theorem 4.12 Let V be an n-dimensional vector space.
Prove it.

(a) If $S = \{v_1, v_2, ..., v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
(b) If $S = \{v_1, v_2, ..., v_n\}$ spans V, then S is a basis for V.

Theorem 4.13 Let S be a finite subset of the vector space V that spans V. A maximal independent subset T of S is a basis for V.
Prove it.

Consider the homogeneous system: $Ax = 0$, where A is an mxn matrix.

Questions:

Which columns of A are linearly independent?
Find a basis for the solution space of A.

The associated augmented matrix $[A|0]$ has reduced row echelon form $[B|0]$, where B has B has r nonzero rows, $1 \leq r \leq m$.
The leading 1's occur in the columns of B, where the corresponding columns of A are linearly independent.
Without loss of generality (by reshuffling the columns of A), assume that the leading 1's in the r nonzero rows occur in the first r columns.

case 1: $r = n$

$$[B|0] = \left[\begin{array}{ccccc|c} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 \end{array}\right]$$

* $Ax = 0$ only has the trivial solution.
* The columns of A are linearly independent.
* If $r < m$ the columns of A do not span $\mathbb{R}^m$.

case 2: $r < n$

$$[B|0] = \left[\begin{array}{cccccccc|c} 1 & 0 & 0 & \dots & 0 & b_{1,r+1} & \dots & b_{1,n} & 0 \\ 0 & 1 & 0 & \dots & 0 & b_{2,r+1} & \dots & b_{2,n} & 0 \\ 0 & 0 & 1 & \dots & 0 & \vdots & & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_{r,r+1} & \dots & b_{r,n} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{array}\right]$$

* $Ax = 0$ has a non trivial solution space of dimension $n - r$. Solve for the unknowns corresponding to the leading 1's.
* The columns of A are linearly dependent.
* If $r < m$ the columns of A do not span $\mathbb{R}^m$.

Problem: Find a basis for the nullspace of $\begin{bmatrix} 1 & -1 & 1 & -2 & 1 \\ 3 & -3 & 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Problem: Find a basis for the solution space of teh homogeneous system: $(\lambda I_n - A)x = 0$, where $\lambda = -3$, $A = \begin{bmatrix} -4 & -3 \\ 2 & 3 \end{bmatrix}$.

Problem: Determine the solution of the linear system $Ax = b$, and write it in the form: $x = x_p + x_h$, where $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$.

Show that if the nxn coefficient matrix A of the homogeneous system $Ax = 0$ has a row (column) of zeros, then $Ax = 0$ has a nontrivial solution.

Definition: Let V be an n-dimensional vector space with ordered basis $S = \{v_1, v_2, ..., v_n\}$. Then a vector $v \in V$ can be uniquely expressed as $v = a_1v_1 + a_2v_2 + ... + a_nv_n$.

Thus $[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ v_n \end{bmatrix}_S$ is the **coordinate vector of v with respect to S.**

Problem: Let $V = M_{22}$, $S = \{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\}$, $v = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$.
Find $[v]_S$.

Definition 4.13 Let $(V, +, *)$ and $(W, \oplus, \odot)$ be real vector spaces. A **one-to-one** function L mapping V onto W is called an **isomorphism** of V onto W if

(a) $L(u + v) = L(u) \oplus L(v)$, for u, v in V;
(b) $L(c * u) = c \odot L(u)$ for u in V, c a real number.

Theorem 4.14 If V is an n-dimensional real vector space, then V is isomorphic to $\mathbb{R}^n$.
Give proof outline.

Theorem 4.15
(a) Every vector space V is isomorphic to itself.
(b) If V is isomorphic to W, then W is isomorphic to V.
(c) If U is isomorphic to V and V is isomorphic to W, then U is isomorphic to W.

Theorem 4.16 Two finite dimensional vector spaces are isomorphic if and only if their dimensions are equal.
Give proof outline.

Corollary 4.6 If V is a finite dimensional vector space that is isomorphic to $\mathbb{R}^n$, then $\dim V = n$.

Definition: Let V be an n-dimensional vector space with ordered bases $S = \{v_1, v_2, ..., v_n\}$, and $T = \{w_1, w_2, ..., w_n\}$.

Then every vector $w_i \in T$ can be uniquely expressed as $w = a_{1i}v_1 + a_{2i}v_2 + ... + a_{ni}v_n$, so $[w_i]_S = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix}_S$.

The nxn, where the i^{th} column is $[w_i]_S$ is called the **transition matrix from the T-basis to the S-basis:** $P_{S \longleftarrow T}$.
Then $[v]_S = P_{S \longleftarrow T}[v]_T$.

Let M_S be the nxn-matrix, whose i^{th} column is v_i, and let M_T be the nxn-matrix, whose i^{th} column is w_i.
Then it can be shown (exercises 39, 40, 41), that $P_{S \longleftarrow T} = M_S^{-1} * M_T$.

Problem: Let $V = \mathbb{R}^2$, $S = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$, $T = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\}$, $v = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, w = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.
Find $[v]_T$, $[w]_T$.

Find $P_{S\longleftarrow T}$.
Find $[v]_S$, $[w]_S$.
Find $[v]_S$ directly.
Find $P_{T\longleftarrow S}$.
Find $[v]_T$, $[w]_T$ using $P_{T\longleftarrow S}$.

Problem: Find an isomorphism $L : \mathbb{P}_2 \longrightarrow \mathbb{R}^3$.
More generally, show that $\mathbb{P}_n$ and $\mathbb{R}^{n+1}$ are isomorphic.

Definition 4.14 Let A be an mxn matrix. The rows of A, considered as vectors in $\mathbb{R}^n$, span a subspace of $\mathbb{R}^n$ called the **row space of** A.
Similarly, the columns of A, considered as vectors in $\mathbb{R}^m$, span a subspace of $\mathbb{R}^m$ called the **column space of** A.

Theorem 4.17 If A and B are two mxn row(column) equivalent matrices, then the row(column) spaces of A and B are equal.
proof outline.

Problem: Find a basis for the row space of A consisting of vectore that are
(a) not necessarily row vectors of A,
(b) row vectors of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 9 & -1 \\ -1 & 8 & 3 \\ -2 & 3 & 2 \end{bmatrix}.$$

Def 4.15 The dimension of the row(column) space of A is called the **row (column) rank**.

Theorem 4.18 The row and column rank of the mxn matrix A are equal.
prove it.

Problem: Find the row and column rank of $A = \begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 2 & -4 & 0 & 1 & 1 \\ 5 & -1 & -3 & 7 & 1 \\ 3 & -9 & 1 & 0 & 2 \end{bmatrix}$

Theorem 4.19 If A is an mxn matrix, then rank $A+$ nullity $A = n$.
prove it.

Note: Equivalent matrices have the same rank. And if two matrices have the same rank, they are equivalent.

Problem: Find the rank of A by obtaining a matrix of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 2 & 3 & 6 & 7 \\ 2 & 1 & 3 & 6 & 5 \end{bmatrix}.$$

Theorem 4.20 If A is an nxn matrix, then rank $A = n$ if and only if A is row equivalent to I_n.

Corrollary 4.7 A is nonsingular if and only if rank $A = n$.

Corrollary 4.8 If A is an nxn matrix, then rank $A = n$ if and only if $det(A) \neq 0$.

Corrollary 4.9 The homogeneous system $Ax = 0$, where A is nxn, has a nontrivial solution if and only if rank $A < n$.

Corrollary 4.10 Let A be an nxn matrix. The linear system $Ax = b$ has a unique solution for every $nx1$ matrix b if and only if rank $A = n$.

Problem: Is this system consistent? $\left[\begin{array}{cccc|c} 1 & -2 & -3 & 4 & 1 \\ 4 & -1 & -5 & 6 & 2 \\ 2 & 3 & 1 & -2 & 2 \end{array}\right]$

Theorem 4.21 The linear system $Ax = b$ has a solution if and only if rank A = rank $[A|b]$, that is if and only if the ranks of the coefficient and augmented matrices are equal.

Prove: Let A be an mxn matrix. Show that the linear system $Ax = b$ has a solution for every $mx1$ matrix b if and only if rank $A = m$.

The following are equivalent for an nxn matrix A:

1. A is nonsingular
2. $Ax = 0$ has only the trivial solution.
3. A is row (column) equivalent to I_n.
4. For every vector b in $\mathbb{R}^n$, the system $Ax = b$ has a unique solution.
5. A is a product of elementary matrices.
6. det $A \neq 0$.
7. The rank of A is n.
8. The nullity of A is zero.
9. The rows of A form a linearly independent set of vectors in $\mathbb{R}^n$.
10. The rows of A form a linearly independent set of vectors in $\mathbb{R}^n$.

Definition: The **length (magnitude)** of a vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ $(\mathbb{R}^3)$, denoted by $||v||$, is

$$||v|| = \sqrt{v_1^2 + v_2^2}.$$

Problem: Find the length of $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \in \mathbb{R}^2$ and of $v = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix} \in \mathbb{R}^3$.

A **unit vector** is a vector of length 1.

Problem: Show, that if x is a nonzero vector in $\mathbb{R}^2$ $(\mathbb{R}^3)$, then $u = \frac{1}{||x||}x$ is a unit vector in direction x.

The **distance** between u and $v \in \mathbb{R}^2$ $(\mathbb{R}^3)$ is $||v - u|| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$

Problem: compute the distance between $v = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix}$.

The **angle** between u and $v \in \mathbb{R}^2$ is: $\cos\theta = \frac{u_1v_1+u_2v_2}{||u||||v||}$, $0 \leq \theta \leq \pi$.

Problem: compute the angle between: $u = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -4 \\ 5 \\ 6 \end{bmatrix}$.

The **standard inner product** on $\mathbb{R}^2$ is the **dot product**, which is a function:
$\mathbb{R}^2 x \mathbb{R}^2 \longrightarrow \mathbb{R}$
$(u, v) \longrightarrow u \cdot v = u_1v_1 + u_2v_2$.

Note: If $u, v \in \mathbb{R}^2$ ($\mathbb{R}^3$):
(i) $||u|| = \sqrt{u \cdot u}$
(ii) for the angle θ between two nonzero vectors u, v: $\cos\theta = \frac{u \cdot v}{||u||||v||}$, $0 \leq \theta \leq \pi$.
(iii) the nonzero vectors u and v are **orthogonal (perpendicular)** if and only if $u \cdot v = 0$.

Theorem 5.1: Let u, v, and w be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let c be a scalar. The standard inner product on $\mathbb{R}^2$ and $\mathbb{R}^3$ has the following properties:
(a) $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$.
(b) $v \cdot u = u \cdot v$
(c) $(u + v) \cdot w = u \cdot w + v \cdot w$
(d) $cu \cdot v = c(u \cdot v)$, for any real scalar c.

Problem: If $x, v, w \in \mathbb{R}^2$ ($\mathbb{R}^2$), and x is orthogonal to both v, w, then x is orthogonal to every vector in span$\{v, w\}$.

Def 5.1 V a real vector space. An **inner product** on V is a function: V x $V \longrightarrow \mathbb{R}$ satisfying:
(i) $(u, u) \geq 0$.
(ii) $(u, v) = (v, u)$ for $u, v \in V$
(iii) $(u + v, w) = (u, w) + (v, w)$, for $u, v, w \in V$
(iv) $(cu, v) = c(u, v)$, for $c \in \mathbb{R}$, $u, v \in V$

Examples:
(a) The dot product on $\mathbb{R}^n$: $(u, v) = u \cdot v$.
(b) On $\mathbb{R}^2$: $(u, v) = u_1v_1 - u_1v_2 - u_2v_1 + 3u_2v_2$.
(c) On $C[0, 1]$ (the space of continuous, real valued functions on [0,1]):
$(f, g) = \int_0^1 f(t)g(t)dt$

Problem: Find the inner product (f, g) for $f(t) = \sin t, g(t) = \cos t \in C[0, 1]$.

Theorem 5.2 Let $S = \{u_1, u_2, ..., u_n\}$ be an ordered basis for a finite dimensional vector space V with an inner product. Let $c_{ij} = (u_i, u_j)$ and $C = [c_{ij}]$. Then
(a) C is a symmetric matrix.
(b) C determines (v, w) for every v and w in V.

prove it

C is the **matrix of the inner product with respect to the basis** S.

Def 5.2 A vector space with an inner product is called an **inner product space**.
If the space is finite dimensional, it is called a **Euclidean space**.

Theorem 5.3 Cauchy - Schwarz Inequality
If u, v are vectors in an inner product space V, then $|(u,v)| \leq ||u|| \cdot ||v||$.
prove it.

Corollary 5.1 Triangle Inequality
If u, v are vectors in an inner product space V, then $||u+v|| \leq ||u|| + ||v||$.
prove it.

Def 5.3 If V is an inner product space, we define the **distance** between two vectors u and v in V as **d(u,v)**$= ||u-v||$.

Def 5.4 Let V be an inner product space. Two nonzero vectors u and v in V are **orthogonal** if $(u,v)=0$.

Def 5.5 Let V be an inner product space. A set S of vectors is called **orthogonal** if any two distinct vectors in S are orthogonal.
If, in addition, each vector in S is of unit length, then S is called **orthonormal**.

Problem: Find the cosine of the angle between $f(t) = \sin t, g(t) = \cos t \in C[0,1]$ under the inner product above.

Theorem 5.4 Let $S = \{u_1, u_2, ..., u_n\}$ be a orthogonal set of nonzero vectors in an inner product space V. Then S is linearly independent.
prove it.

Problem: Let V be an inner product space. Show that if v is orthogonal to $w_1, w_2, ..., w_k$, then v is orthogonal to every vector in span$\{w_1, w_2, ..., w_k\}$.

Theorem 5.5 Let $S = \{u_1, u_2, ..., u_n\}$ be an orthonormal basis for a Euclidean space V and let v be any vector in V. Then $v = c_1u_1 + c_2u_2 + ... + c_nu_n$, where $c_i = (v, u_i)$, $i = 1, 2, ..., n$.

Theorem 5.6 Gram-Schmidt Process
Let V be an inner product space and $W \cdot \{0\}$ an m-dimensional subspace of V.
Then there exists an ortonormal basis $T = \{w_1, w_2, ..., w_m\}$ for W.

Problem: Find an orthonormal basis for the subspace of $\mathbb{R}_4$ consisting of all vectors of the form $\begin{bmatrix} a & a+b & c & b+c \end{bmatrix}$.

Theorem 5.7 Let V be an n-dimensional Euclidean space, and let $S = \{u_1, u_2, ..., u_n\}$ be an orthonormal basis for V.

If $v = a_1u_1 + a_2u_2 + ... + a_nu_n$ and $w = c_1u_1 + c_2u_2 + ... + c_nu_n$, then

$$(v, w) = v = a_1b_1 + a_2b_2 + ... + a_nb_n.$$

prove it.

Theorem 5.8 QR Factorization
If A is an mxm matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an mxn matrix whose columns form an orthonormal basis for the column space of A and R is an nxn nonsingular upper triangular matrix.
proof idea.

Problem: Find the QR factorization of $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \\ 0 & 1 & -2 \end{bmatrix}$.

Def 5.6 Let W be a subspace of an inner product space V. A vector u in V is **orthogonal** to W, if it is orthogonal to every vector in W.
The set of all vectors in V that are orthogonal to all vectors in W is called the orthogonal complement of W in V ($W^\perp$).

Theorem 5.9 Let W be a subspace of an inner product space V. Then:
(a) $W^\perp$ is a subspace of V.
(b) $W \cap W^\perp = \{0\}$.

Problem: Show that if W is a subspace of an inner product space V, that is spanned by a set of vectors S, then a vector u in V belongs to $W^\perp$ if and only if u is orthogonal to every vector in S.

Def: Let W_1, W_2 be subspaces of vectorspace V. The V is the **direct sum** of W_1 and W_2:
$V = W_1 \oplus W_2$ iff
(i) $W_1 \cap W_2 = 0$ and
(ii) every vector $v \in V$ can be expressed as $v = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$.

Theorem 5.10 Let W be a finite dimensional subspace of an inner product space V. Then

$$V = W \oplus W^\perp.$$

prove it.

Problem: Let W be a subspace of $\mathbb{R}^4$ spanned by w_1, w_2, w_3, w_4,
$w_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$, $w_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -5 \end{bmatrix}$, $w_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ -2 \end{bmatrix}$, $w_4 = \begin{bmatrix} 7 \\ 2 \\ -1 \\ 4 \end{bmatrix}$.
Find a basis for $W^\perp$.

Theorem 5.11 If W is a finite dimensional subspace of an inner product space V, then $(W^\perp)^\perp = W$.

Theorem 5.12 If $A$is a given mxn matrix, then

(a) The null space of A is the orthogonal complement of the row space of A.

(b) The null space of A^T is the orthogonal complement of the column space of A.

prove it.

Problem: Compute the four fundamental spaces associated with $A = \begin{bmatrix} 1 & 5 & 3 & 7 \\ 2 & 0 & -4 & -6 \\ 4 & 7 & -1 & 2 \end{bmatrix}$.

Note: let $\{w_1, w_2, ..., w_n\}$ be a basis for a subspace W of an inner product space V. Then the orthogonal projection onto W of vector v in V is:

$$proj_w v = [(v, w_1)/(w_1, w_1)]w_1 + [(v, w_2)/(w_2, w_2)]w_2 + ... + [(v, w_n)/(w_n, w_n)]w_n.$$

Problem: Let V be the Euclidean space $\mathbb{R}^4$, and W the subspace with basis $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}$. Find $proj_W v$ for $v = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$.

Theorem 5.13 Let W be a finite dimensional subspace of the inner product space V.
Then, for vector v belonging to V, the vector in W closest to v is $proj_w v$. That is, $||v - w||$, for w belonging to W, is minimized by $w = proj_w v$.
prove it.

Fourier Analysis:
Taylor and McLaurin expansions may not exist in some cases, such as when f is not differentiable at some point a, or f may not even be continuous.
However, there is another expansion.
Check that on $[-\pi, \pi]$ with the inner product $(f, g) = \int_{-\pi}^{\pi} fgdt$ the following functions form an orthonormal set S:
$\frac{1}{2\pi}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \cos 2t, \frac{1}{\sqrt{\pi}} \sin 2t, ...$

For any function f, we can project it onto the finite subspace spanned by the first n vectors of S: which results in the **Fourier polynomial of degree n for f**.

Theorem 5.14 If A is an mxn matrix with rank n, then $A^T A$ is nonsingular and the linear system $Ax = b$ has a unique least squares solution given by

$$x = (A^T A)^{-1} A^T b.$$

Problem: Determine the least squares solution to $Ax = b$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \\ -2 \end{bmatrix}$.

Def 6.1 Let V, W be vector spaces. A function $L : V \longrightarrow W$ is a **linear transformation** of V into W if for every u, v in V and real number c:

(a) $L(u+v) = L(u) + L(v)$,
(b) $L(cu) = cL(u)$
If $V = W$ then L is also called a linear operator.

Examples: reflection, projection, dilation, contraction, rotation.

Problem: Which of the following functions is a linear transformation?
(a) $L : \mathbb{R}_2 \longrightarrow \mathbb{R}_3$
$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \longrightarrow \begin{bmatrix} u_1+1 & u_2 & u_1+u_2 \end{bmatrix}$
(b) $L : \mathbb{R}_2 \longrightarrow \mathbb{R}_3$
$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \longrightarrow \begin{bmatrix} u_1+u_2 & u_2 & u_1-u_2 \end{bmatrix}$

Theorem 6.1 Let $L : V \longrightarrow W$ be a linear transformation. Then
(a) $L(0_v) = 0_w$.
(b) $L(u-v) = L(u) - L(v)$, for $u, v \in V$.

Theorem 6.2 Let $L : V \longrightarrow W$ be a linear transformation of an n-dimensional vector space V into a vectorspace W. Let $S = \{v_1, v_2, ..., v_n\}$ be a basis for V. If v is any vector in V, then $L(v)$ is completely determined by $\{L(v_1), L(v_2), ..., L(v_n)\}$.

Problem: Let W be the vector space of all real valued functions and let Vbe the subspace of all differentiable functions. Define $L : V \longrightarrow W$, $L(f) = f'$, where f' is the drivative of f. Prove that L is a linear transformation.

Theorem 6.3 Let L: $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and consider the natural basis $\{e_1, e_2, ..., e_n\}$ for $\mathbb{R}^n$. Let A be the mxn matrix whose j'th column is $L(e_2)$. The matrix A has the following property: If $x = [x_1 x_2 ... x_n]^T$ is any vector in $\mathbb{R}^n$, then $L(x) = Ax$.
Moreover, A is the only matrix satisfying equation (1). It is called the **standard matrix representing L**.

Definition 6.2 A linear transformation is called **one-to-one**, if $L(u) = L(v)$ implies $u = v$.

Definition 6.3 Let $L : V \longrightarrow W$ be a linear transformation of a vector space V into a vector space W. The **kernel of L**, **ker L**, is the subset of V consisting of all v of V such that $L(v) = 0$.

Theorem 6.4 Let $L : V \longrightarrow W$ be a linear transformation of a vector space V into a vectorspace W. Then
(a) ker L is a subspace of V.
(b) L is one-to-one if and only if ker $L = \{0\}$.

Corollary 6.1 If $L(x) = b$ and $L(y) = b$, then $x - y$ belongs to ker L, i.e. any two solutions to $L(x) = b$ differ by an element of the kernel of L.

Def 6.4 Let $L : V \longrightarrow W$ be a linear transformation of a vector space V into a vectorspace W, then the **range or image** of V under L, denoted by **range** L, consists of those vectors in W that are images under L of some vector in V.

w is in the range L iff there exists a vector $v \in V$ such that L(v) = w.

L is called **onto** if im $L = W$.

Theorem 6.5 Let $L : V \longrightarrow W$ be a linear transformation of a vector space V into a vectorspace W, then range L is a subspace of W.

Problem: Let $L : M_{23} \longrightarrow M_{33}$ be the linear transformation defined by $L(A) = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} A$, for $A \in M_{23}$.

(a) Find the dimension of ker L.
(b) Find the dimension of range L.

Theorem 6.6 Let $L : V \longrightarrow W$ be a linear transformation of an n-dimensional vector space V into a vector space W, then

$$\dim \ker L + \dim \text{range } L = \dim V.$$

prove it

Problem: Let $L : M_{22} \longrightarrow M_{22}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a+b & b+c \\ a+d & b+d \end{bmatrix}$.

(a) Find a basis for ker L.
(b) Find a basis for range L.
(c) Verify Theorem 6.6 for L.

Corollary 6.2 Let $L : V \longrightarrow W$ be a linear transformation of a vector space V into a vector space W, and dim V = dim W, then

(a) If L is one-to-one, then L is onto.
(b) If L is onto, then L is one-to-one.

Def A linear transformation $L : V \longrightarrow W$ if a vector space V to a vector space W is **invertible** if it is an invertible function, i.e. if there a unique function L: $W \longrightarrow V$ such that

$L \circ L^{-1} = I_w$ and
$L^{-1} \circ L = I_v$,

where I_v = identity on V and I_w = identity on W.

Theorem 6.7 A linear transformation $L : V \longrightarrow W$ is invertible if and only if $L$is one-to-one and onto.
Moreover, L^{-1} is a linear transformation and $(L^{-1})^{-1} = L$.
give a proof outline.

Problem: Let $L : C[a, b] \longrightarrow \mathbb{R}$ be the linear transformation $L(f) = \int_a^b f(x)dx$. Show that L is not 1-1, but onto.

Theorem 6.8 A linear transformation L: $V \longrightarrow W$ is one-to-one if and only if the image of every linearly in dependent set of vectors is a linearly independent set of vectors.

Theorem 6.9 Let L: $V \longrightarrow W$ be a linear transformation of an n-dimensional vector space V into an m-dimensional vector space W ($n \neq 0$, $m \neq 0$) and
let $S = \{v_1, v_2, ..., v_n\}$ and $T = \{w_1, w_2, ..., w_m\}$ be ordered bases for V and W, respectively.
Then the mxn matrix A whose j'th column is the coordinate vector $[L(v_j)]_T$ of $L(v_j)$ with respect to T has the following property:
$[L(v_j)]_T = A[x]_S$ for every x in V.

Problem: Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \begin{bmatrix} a+2b \\ 2a-b \end{bmatrix}$

Let S be the natural basis, $T = \{\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}\}$.

Find the representation of L with respect to
(a) S (b) S and T (c) T and S (d) T

Theorem 6.10 Let U be the vector space of all linear transformations of an n-dimensional vector space V into an m-dimensional vector space W, $n \neq 0$ and $m \neq 0$, under the operations + and *. Then U is isomorphic to the vector space M_{mn} of all mxn matrices.

Problem: Let $L : M_{22} \longrightarrow M_{22}$, $L(X) = AX - XA$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Let S be the standard basis for M_{22}, $T = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\}$.

Find the representation of L with respect to (a) S (b) S and T (c) T and S (d) T

Theorem 6.11 Let V_1 be an n-dimensional vector space,
V_2 be an m-dimensional vector space,
and V_3 a p-dimensional vector space
with linear transformantions L_1 and L_2 such that
$L_1 : V_1 \longrightarrow V_2$,
$L_2 : V_2 \longrightarrow V_3$.
If the ordered bases P, S, and T are chosen for V_1, V_2, and V_3, respectively,
then $M(L_1 \circ L_2) = M(L_1)M(L_2)$.

Theorem 6.12 Let $L : V \longrightarrow W$ be a linear transformationof an n-dimensional vector space V into an m-dimensional vector space W.
Let $S = \{v_1, v_2, ..., v_n\}$, $S' = \{v_1', v_2', ..., v_n'\}$ be ordered bases for V,
and $T = \{w_1, w_2, ..., w_m\}$ and $T' = \{w_1', w_2', ..., w_m'\}$ be ordered bases for W.
Let $P_{S \leftarrow S'}$, $P_{T' \leftarrow T}$ be transition matrices.
Let ${}_TA_S$ be the representation of L with respect to S and T,
then the representation ${}_{T'}A_{S'}$ of L with repsect to S' and T' is

$${}_{T'}A_{S'} = P_{T' \leftarrow T} *_T A_S * P_{S \leftarrow S'}$$

. prove it.

Problem: Let $L : P_1 \longrightarrow P_2$, $L(p(t)) = t * p(t) + p(0)$
P_1 has bases $S = \{t, 1\}, S' = \{t+1, t-1\}$.
P_2 has bases $T = \{t^2, t, 1\}, T'' = \{t^2+1, t-1, t+1\}$.
Find ${}_T L_S$, ${}_{T'} L_S$, ${}_T L'_S$, ${}_{T'} L_{S'}$.

Corollary 6.3 Let $L : V \longrightarrow V$ be a linear operator of an n-dimensional vector space V.
Let $S = \{v_1, v_2, ..., v_n\}$, $S' = \{v'_1, v'_2, ..., v'_n\}$ be ordered bases for V.
Let $P = P_{S \leftarrow S'}$ be the transition matrix.
Let ${}_S A_S$ be the representation of L with respect to S ,
then the representation ${}_{S'} A_{S'}$ of L with respect to S' is

$${}_{S'} A_{S'} = P_{S' \leftarrow S} *_S A_S * P_{S \leftarrow S'} = P^{-1} * A * P.$$

Theorem 6.13 Let L: $V \longrightarrow W$ be a linear transformation. Then rank L= dim range L.

Definition 6.6 If A and B are nxn matrices, then B is **similar** to A if there is a nonsingular P such that $B = P^{-1}AP$.

Problem: Let A, B, C be square matrices. Show that
(a) A is aimilar to A.
(b) If A is similar to B, then B is similar to A.
(c) If A is similar to B and B is similar to C, then A is similar to C.

So similarity is an equivalence relation.

Theorem 6.14 Let V be any n-dimensional vector space and let A and B be any nxn matrices. Then A and B are similar if and only if A and B represent the same linear transformation $L : V \longrightarrow V$ with respect to two ordered bases for V.

Theorem 6.15 If A and B are similar nxn matrices, then rank A= rank B.

Problem: prove, that if A and B are similar, then A^T and B^T are similar.

Problem: prove, that if A and B are similar, then Tr(A) = Tr(B).

Problem: Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation whose representation with respect to the natural basis is $A = [a_{ij}]$.

Let $P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Find a basis T of $\mathbb{R}^3$ with respect to which $B = P^{-1}AP$ represents L.

Definition 7.1 Let L: $V \longrightarrow V$ be a linear transformation of an n-dimensional vector space into itself. The number λ is called an **eigenvalue** of L if there exists a nonzero vector $x \in V$ such that $L(x) = \lambda x$. Every nonzero vector x satisfying this equation is then called an **eigenvector** of L associated with the eigenvalue λ.

Problem: Find the eigenvalues and eigenvectors of $\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$.

Definition 7.2 Let $A = [a_{ij}]$ be an nxn matrix. Then the determinant of the matrix

$$\lambda I_n - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{bmatrix}$$

is called the **characteristic polynomial** of A.

The equation $p(\lambda) = det(\lambda I_n - A) = 0$ is called the **characteristic equation** of A.

Problem: Find the eigenvalues and eigenvectors of $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Theorem 7.1 Let A be an nxn matrix. The eigenvalues of A are the roots of the characteristic polynomial of A.
prove it.

Problem: Prove that A and A^T have the same eigenvalues.

Problem: Let $L : V \longrightarrow V$ be an invertible linear operator and let λ be an eigenvalue of L with associated eigenvector x. Show, that $\frac{1}{\lambda}$ is an eigenvalue of L^{-1} with associated eigenvector x.

Problem: Show, that if A is a matrix all of whose columns add up to 1, then $\lambda = 1$ is an eigenvalue of A. (Hint: consider the product $A^T x$, where x is the vector all of whose entries are 1, and use a previous exercise.)

Definition 7.3 Let L: $V \longrightarrow V$ be a linear transformation of an n-dimensional vector space into itself. We say that L is diagonalizable, or can be **diagonalized**, if there exists a basis S for V such that L is represented with respect to S by a diagonal matrix D.

Theorem 7.2 Similar matrices have the same eigenvalues.
prove it.

Theorem 7.3 Let L: $V \longrightarrow V$ be a linear transformation of an n-dimensional vector space into itself.
Then L is diagonalizable if and only if V has a basis S of eigenvectors of V.
Moreover, if D is a diagonal matrix representing L with respect to S, then the entries on the main diagonal are the eigenvalues of L.

Problem: For the matrix A find, if possible, a nonsingular matrix P, so that $P^{-1}AP$ is diagonal.
$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Theorem 7.4 An nxn matrix A is similar to a diagonal matrix D if and only if A has n linearly independent eigenvectors.
Moreover, the elements on the main diagonal of D are the eigenvalues of A.

Problem: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, can A be diagonalized?

Theorem 7.5 If the roots of the characteristic polynomial of an nxn matrix A are all different from each other (i.e., distinct), then A is diagonalizable.
prove it.

Problem: Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$. Compute A^9. (Hint: Find a matrix P so that $P^{-1}AP$ is a diagonal matrix D and show that $A^9 = PD^9P^{-1}$.)

Problem: Let A, B be nonsingular matrices. Prove that AB and BA have the same eigenvalues.

Problem: Prove that if A is diagonalizable, then A^T is diagonalizable.

Problem: Diagonalize $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Theorem 7.6 All roots of the characteristic polynomial of a symmetric matrix are real numbers.
prove it.

Theorem 7.7 If A is a symmetric matrix, then the eigenvectors that belong to distinct eigenvalues of A are orthogonal.
prove it.

Definition 7.4 A real square matrix A is called **orthogonal** if $A^T = A^{-1}$, i.e. if $A^TA = I_n$.

Problem: If A, B are orthogonal matrices, then AB is an orthogonal matrix.

Problem: If A is an orthogonal matrix, then the determinant of A is 1 or -1.

Theorem 7.8 The nxn matrix A is orthogonal if and only if the columns (rows) of A form an orthonormal set.
did this on exam 2.

Theorem 7.9 If A is a symmetric nxn matrix, then there exists an orthogonal matrix P such that

$$P^{-1}AP = P^TAP = D.$$

The eigenvalues of A lie on the main diagonal of D.
for a proof of the fact, that for a symmetric matrix with eigenvalue λ of multiplicity m, there is a full set of m linearly independent eigenvectors, see Ortega: Matrix Theory: A Second Course.

Problem: Let $L\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear operator, $L(x) = Ax$, then the following statements are equivalent:
(i) A orthogonal
(ii) $(L(x), L(y)) = (x, y)$, i.e.L preserves the angle between x and y.
Such an L is called an **isometry**.

Problem: Let $p_1(\lambda)$ be the characteristic polynomial of A_{11} and $p_2(\lambda)$ the characteristic polynomial of A_{22}. What is the characteristic polynomial of
(a) $A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ and (b) $A' = \begin{bmatrix} A_{11} & A_{21} \\ 0 & A_{22} \end{bmatrix}$?

Problem: Prove or disprove: If we interchange two rows in a square matrix, then the eigenvalues are unchanged.

Problem: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Is it true that if A and B have the same trace, determinant and eigenvalues, then they are similar?

Problem: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(i) Find the eigenvalues and eigenvectors of A.
(ii) Find the diagonal matrix similar to A.
(iii) How are A and D related to the Fibonacci numbers?

Problem: Let A be an nxn real matrix. Show that the trace of A ($\mathrm{tr}(A)$) is the sum of the eigenvalues of A.