Problem 1.7.1: Evaluate $\int x^n \ln x \, \mathrm{d}x$ where $n \neq 1$.

Solution Integrate by parts, letting $u = \ln x$ and $\mathrm{d}v = x^n \mathrm{d}x$. With these choices, $\mathrm{d}u = \frac{1}{x}\,\mathrm{dx}$ and $v = \frac{x^{n+1}}{n+1}$. The integral becomes $uv - \int v \mathrm{d}u$, giving us

$$\frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}\,\mathrm{d}x}{x}$$

$$\frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\,\mathrm{d}x$$

$$\frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$$

Problem 1.7.3: Evaluate $\int e^{ax}\cos bx\,\mathrm{d}x$ where $a, b \neq 0$.

Solution Let $u = \cos bx$ and $\mathrm{d}v = e^{ax}\mathrm{d}x$. Then $\mathrm{d}u = -b\sin bx \mathrm{d}x$ and $v = \frac{1}{a}e^{ax}$. Integrating by parts, we have

$$\int e^{ax}\cos bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx - \frac{-b}{a}\int e^{ax}\sin bx\,\mathrm{d}x.$$

We can handle the integral on the right in a similar way. Let $u = \sin bx$ and $\mathrm{d}v = e^{ax}\mathrm{d}x$. Then $\mathrm{d}u = b\cos bx \mathrm{d}x$ and $v = \frac{1}{a}e^{ax}$. This gives us

$$\int e^{ax}\cos bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left(\frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx\,\mathrm{d}x\right) + C$$

Notice that the integral we started with has reappeared on the right-hand side of the equation. If we distribute the $\frac{b}{a}$ and add this term to both sides, we get

$$\left(1 + \frac{b^2}{a^2}\right)\int e^{ax}\cos bx\,\mathrm{d}x = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx + C$$

$$\therefore \int e^{ax}\cos bx\,\mathrm{d}x = \frac{\frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx}{1 + \frac{b^2}{a^2}} + C$$

Problem 1.7.5: Use §6.3 to evaluate $\sin^2 x\,\mathrm{d}x$. Show that the answer is the same as the answer you get using the half-angle formula.

Solution The reduction formula is

$$\int \sin^n x\,\mathrm{d}x = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^n x\,\mathrm{d}x$$

We are dealing with the case $n = 2$. This gives us

$$\int \sin^2 x\,\mathrm{d}x = -\frac{1}{2}\sin x\cos x + \frac{2-1}{2}\int \sin^0 x\,\mathrm{d}x$$

Since $\sin^0 x = 1$, this is simply

$$\int \sin^2 x \, \mathrm{d}x = -\frac{\sin x \cos x}{2} + \frac{x}{2} + C$$

The identities $\sin^2 x = \frac{1-\cos(2x)}{2}$ and $\sin(2x) = 2\sin x \cos x$ let you derive this answer in a different way.

Problem 1.7.7: Investigate the numbers $A_n = \int_0^\pi \sin^n x \, \mathrm{d}x$.

Solution For part (a), note that $A_0 = \int_0^\pi \sin^0 x \, \mathrm{d}x = \int_0^\pi \mathrm{d}x = \pi - 0 = 0$. Moreover, $\int_0^\pi \sin x \, \mathrm{d}x = (-\cos x) \, |_0^\pi = 2$.

For part (b), we use the reduction formula. The reduction formula tells us that

$$I_5 = -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5} I_3$$

and also that

$$I_3 = -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3} I_1.$$

It follows that

$$I_5 = -\frac{1}{5}\sin^4 x \cos x + \frac{4}{5}\left(-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}(-\cos x)\right)$$

Evaluating this at the endpoints 0 and π gives us $A_5 = \frac{4}{5}\frac{2}{3}(2) = \frac{16}{15}$. The process for A_6 and A_7 is similar. You should find that $A_6 = \frac{5\pi}{16}$ and $A_7 = \frac{6}{7}\frac{16}{15} = \frac{32}{35}$.

In part (c), note that $\sin x$ is non-negative over the interval $[0, \pi]$. Because $\sin x \leq 1$, it follows that $\sin^n x \leq \sin^{n-1} x$. (To see why, think about what happens with $(\frac{1}{2}), (\frac{1}{2})^2$, etc.) Since the height of the function $\sin^n x$ is less than that of $\sin^{n-1} x$ over this interval, the overall area should be smaller (here we are using the fact that $\sin^n x$ is *strictly* smaller than $\sin^{n-1} x$ at most points in this interval; if they were equal everywhere, we couldn't make the same conclusion about the areas). Hence $A_n < A_{n-1}$, as claimed.

For part (d), notice that $A_7 < A_6 < A_5$ by part (c). Hence $\frac{32}{35} < \frac{5\pi}{16} < \frac{16}{15}$. Multiplying through by $\frac{16}{15}$, we see that

$$\frac{512}{175} < \pi < \frac{256}{75}$$

Problem 1.7.9: Prove the formula $\int x^m (\ln x)^n \, \mathrm{d}x = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} \, \mathrm{d}x$ for $m \neq -1$.

Solution We integrate by parts. Let $u = (\ln x)^n$ and $\mathrm{d}v = x^m \mathrm{d}x$. Then $\mathrm{d}u = \frac{n(\ln x)^{n-1}}{x}\mathrm{d}x$ by the chain rule and $v = \frac{1}{m+1}x^{m+1}$. This gives us

$$\int x^m (\ln x)^n \, \mathrm{d}x = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^{m+1}\frac{(\ln x)^{n-1}}{x} \, \mathrm{d}x.$$

Notice that $\frac{x^{m+1}}{x} = x^m$, verifying the reduction formula.

Problem 1.7.11: Find $\int (\ln x)^2 \, \mathrm{d}x$.

Solution This is simply the case of the formula from the previous problem when $m = 0$ and $n = 2$. Plugging these values in, we have

$$\int (\ln x)^2 \, \mathrm{d}x = x(\ln x)^2 - 2 \int \ln x \, \mathrm{d}x$$

In class we saw that $\int \ln x \, \mathrm{d}x = x \ln x - x + C$ by using integration by parts. Combining these, we have

$$\int (\ln x)^2 \, \mathrm{d}x = x(\ln x)^2 - 2(x \ln x - x) + C.$$

Problem 1.7.13: Evaluate $\int x^{-1} \ln x \, \mathrm{d}x$ by another method.

Solution A simple substitution works. Let $u = \ln x$. Then $\mathrm{d}u = x^{-1}\mathrm{d}x$, so we are left with $\int u \, \mathrm{d}u = \frac{u^2}{2} + C$. In terms of x, we get $\frac{(\ln x)^2}{2} + C$.

Problem 1.7.15: Find $\int \frac{\mathrm{d}x}{(1+x^2)^3}$.

Solution This calculation is done on page 18 of the course notes.

Problem 1.7.17: Find $\int \frac{x\mathrm{d}x}{(1+x^2)^4}$.

Solution Instead of using the reduction formula, we perform a substitution. Let $u = 1+x^2$. Then $\mathrm{d}u = 2x\mathrm{d}x$. The integral in question becomes

$$\frac{1}{2} \int u^{-4} \mathrm{d}u = -\frac{1}{6} u^{-3} + C.$$

In terms of x, we have

$$\int \frac{x\mathrm{d}x}{(1+x^2)^4} = -\frac{1}{6} \frac{1}{(1+x^2)^3} + C$$

Problem 1.7.18: Find $\int \frac{\mathrm{d}x}{(49+x^2)^3}$.

Solution We'd like to transform the integral to be able to apply the relevant reduction formula. To do so, we must remove the pesky 49 by dividing numerator and denominator by 49^3:

$$\int \frac{\frac{\mathrm{d}x}{49^3}}{\left(\frac{49+x^2}{49}\right)^3} = \frac{1}{49^3} \int \frac{\mathrm{d}x}{\left(1 + \left(\frac{x}{7}\right)^2\right)^3}$$

Next, we set $u = \frac{x}{7}$, giving $\mathrm{d}u = \frac{\mathrm{d}x}{7}$. Noticing that $49^3 = 7^6$, this transforms the integral into

$$\frac{1}{7^5} \int \frac{\mathrm{d}u}{(1+u^2)^3}.$$

We evaluated this integral in a previous problem (as well as on page 18 of the course notes). All that remains is to substitute $\frac{x}{7}$ for u.

Problem 1.7.20: Given that the reduction formula

$$I_{n+1} = \frac{1}{2n}\frac{x}{(1+x^2)^n} + \frac{2n-1}{2n}I_n$$

holds for all positive values of n, find a relationship between $I_{1/2} = \int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$ and $I_{-1/2} = \int \sqrt{1+x^2}\,\mathrm{d}x$.

Solution Following the hint, we set $n = -\frac{1}{2}$ in the reduction formula, yielding

$$I_{1/2} = -x\sqrt{1+x^2} + 2I_{-1/2}$$

$$\therefore \int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = -x\sqrt{1+x^2} + 2\int \sqrt{1+x^2}\,dx$$

Problem 1.7.21: Using integration by parts, we arrive at $\int \frac{\mathrm{d}x}{x} = 1 + \int \frac{\mathrm{d}x}{x}$. Subtracting the integral from both sides seems to result in the impossible equality $0 = 1$. What gives?

Solution Throughout the course of the argument, the integrals are *indefinite*, and so are determined only up to an arbitrary constant $+C$. Certainly the equality $0 = 1 + C$ holds for some constant C, thereby resolving the issue.

Problem 1.9.1: Express each of the following rational functions as the sum of a polynomial with a proper rational function.

Solution Using polynomial long division, we find that $\frac{x^3}{x^3-4} = 1 + \frac{4}{x^3-4}$ and that $\frac{x^3-1}{x^2-1} = x + \frac{x-1}{x^2-1} = x + \frac{1}{x+1}$

Problem 1.9.2: Compute the following integrals by completing the square.

Solution (a) Note that $\frac{1}{x^2+6x+8} = \frac{1}{(x+3)^2-1}$. The denominator is the difference of two squares, and so it equals $\frac{1}{(x+3-1)(x+3+1)} = \frac{1}{(x+2)(x+4)}$. Using partial fractions, we set this equal to $\frac{A}{x+2} + \frac{B}{x+4}$ for some constants A, B. Multiplying through by the denominator $(x+2)(x+4)$ gives $1 = A(x+4) + B(x+2)$. Equating coefficients, we have $A + B = 0$ and $4A + 2B = 1$. The solutions to this system of equations are $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Finally, after integrating we are left with

$$\frac{1}{2}\ln|x+2| - \frac{1}{2}\ln|x+4| + C$$

(c) Note that $\frac{1}{5x^2+20x+25} = \frac{1}{5}\frac{1}{x^2+4x+5} = \frac{1}{5}\frac{1}{(x+2)^2+1}$. Let $u = x + 2$. Then $\mathrm{d}u = \mathrm{d}x$. Under this substitution,

$$\int \frac{\mathrm{d}x}{5x^2+20x+25} = \frac{1}{5}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{5}\arctan(x+2) + C$$

Problem 1.9.3: Evaluate $\int \frac{x^2+3}{x(x+1)(x-1)}\,\mathrm{d}x$.

Solution First, we find constants A, B, and C satisfying $\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$. Clearing denominators gives us $Ax^2 + Bx^2 + Cx^2 - Bx + Cx - A = x^2 + 3$. Equating coefficients, we see that $A + B + C = 1$, $-B + C = 0$, and $-A = 3$. Solving these equations gives us $A = -3$, $B = 2$, and $C = 2$. Integrating this is easy:

$$-3\ln|x| + 2\ln|x+1| + 2\ln|x-1| + C$$

Problem 1.9.5: Find the integral $\int \frac{x^2+3}{x^2(x-1)}$.

Solution As before, we first find constants A, B, and C satisfying $\frac{x^2+3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$. Clearing denominators gives us $Ax^2 - Ax + Bx - B + Cx^2 = x^2 + 3$. Equating coefficients, we see that $A + C = 1$, $-A + B = 0$, and $-B = 3$. Solving these equations gives us $A = -3$, $B = -3$, and $C = 4$. Again, once we know the constants the integral is straightforward:

$$-3\ln|x| + \frac{3}{x} + 4\ln|x-1| + C$$

Problem 1.9.6: Simplicio tried so hard and got so far // but in the end, it didn't even matter.

Solution Notice that Simplicio tried to apply partial fractions to a rational function that wasn't proper: the degree of the numerator wasn't strictly smaller than that of the denominator. In fact, suppose we could find A and B satisfying $\frac{4x^2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{(x+1)}$. Clearing the denominators, we'd find that $4x^2 = Ax + A + Bx - 3B$. The right-hand side has no squared terms, while the left-hand side does. This phenomenon explains why we can only apply the method of partial fractions to proper rational functions.

Problem 1.9.7: Find $\int_{-5}^{-2} \frac{x^4-1}{x^2+1}\,\mathrm{d}x$.

Solution Using polynomial long division, we find that the integrand equals $x^2 - 1$. $\int_{-5}^{-2} x^2 - 1\,\mathrm{d}x = (\frac{x^3}{3} - x)\,|_{-5}^{-2} = 36$.

Problem 1.9.9: Find $\int \frac{x^5\mathrm{d}x}{x^2-1}$.

Solution Using polynomial long division, we find that the integrand equals $x^3 + x + \frac{x}{x^2-1}$. We can integrate each of these terms individually, using the substitution $u = x^2 - 1$ in the third term. Doing so yields $\frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{2}\ln|x^2 - 1| + C$.

Problem 1.9.14: Find $\int \frac{e^{3x}\mathrm{d}x}{e^{4x}-1}$.

Solution We begin with the substitution $u = e^x$. Then $\mathrm{d}u = e^x\mathrm{d}x$, so $\mathrm{d}x = \frac{\mathrm{d}u}{u}$. This transforms the integral into $\int \frac{u^2\mathrm{d}u}{u^4-1}$. We can factor the denominator as $(u^2+1)(u+1)(u-1)$, applying the difference of two squares twice. We'll try to write the integrand as $\frac{A}{u+1} + \frac{B}{u+1} + \frac{Cu+D}{u^2+1}$. Clearing the denominator, we get $u^2 = A(u-1)(u^2+1) + B(u+1)(u^2+1) + (Cu + D)(u^2 - 1)$. Expanding and equating coefficients gives the four equations $0 = A + B + C$, $1 = -A + B + D$, $0 = A + B - C$, and $0 = -A + B - D$. This system has the single solution $A = -\frac{1}{4}$, $B = \frac{1}{4}$, $C = 0$, and $D = \frac{1}{2}$. After integrating and substituting $u = e^x$, we arrive at the answer $-\frac{1}{4}\ln|e^x + 1| + \frac{1}{4}\ln|e^x - 1| + \frac{1}{2}\arctan(e^x + 1) + C$.

Problem 1.9.15: Find $\int \frac{e^x \mathrm{d}x}{\sqrt{1+e^{2x}}}$.

Solution This problem seems exceptionally difficult for this section of the book. Let's see what happens if we try the substitution $u = e^x$. Then $\mathrm{d}u = e^x \mathrm{d}x$, so the integral becomes $\int \frac{\mathrm{d}u}{\sqrt{1+u^2}}$. We'll learn how to do this integral later on in this chapter.

Problem 1.9.18: Find $\int \frac{\mathrm{d}x}{x(x^2+1)}$.

Solution We rewrite $\frac{1}{x(x^2+1)}$ as $\frac{A}{x} + \frac{Bx+C}{x^2+1}$ for some constants A, B, and C. Clearing denominators, we find that $1 = A(x^2+1) + (Bx+C)x$. Equating coefficients gives us $A = 1$, $A + B = 0$, and $C = 0$. From this system of equations, we conclude that $A = 1$, $B = -1$, and $C = 0$. Integrating yields $\ln|x| - \frac{1}{2}\ln|x^2+1| + C$, where we have used the substitution $u = 1 + x^2$ to find the second integral.

Problem 1.9.21: Find $\int \frac{1}{(x-1)(x-2)(x-3)} \mathrm{d}x$.

Solution We first rewrite $\frac{1}{(x-1)(x-2)(x-3)}$ as $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$ for some constants A, B, and C. Clearing denominators, we get $A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = 1$. Setting $x = 1$, we find that $2A = 1$, so $A = \frac{1}{2}$. Setting $x = 2$, we find that $B(-1) = 1$, so $B = -1$. Finally, setting $x = 3$ gives us $2C = 1$, so $C = \frac{1}{2}$. (You could also multiply these terms out and equate coefficients, but the Heaviside trick is faster in this case.) Integrating results in the answer $\frac{1}{2}\ln|x-1| - \ln|x-2| + \frac{1}{2}\ln|x-3| + C$.

Problem 1.9.22: $\int \frac{x^2+1}{(x-1)(x-2)(x-3)} \mathrm{d}x$

Solution The procedure is exactly as in the problem above. We get $A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = x^2+1$. Setting $x = 1$, we get $2A = 2$, so $A = 1$. Setting $x = 2$, we get $-B = 5$, so $B = -5$. Finally, setting $x = 3$ gives us $2C = 10$, so $C = 5$. Integating results in the answer $\ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$.

Problem 1.9.23: $\int \frac{x^3+1}{(x-1)(x-2)(x-3)} \mathrm{d}x$

Solution Since the integrand is not a proper rational function, we have to perform polynomial long division before attempting to use partial fractions. Note that $(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$ and that $\frac{x^3+1}{x^3-6x^2+11x-6} = 1 + \frac{6x^2-11x+7}{x^3-6x^2+11x-6}$. We set $\frac{6x^2-11x+7}{x^3-6x^2+11x-6}$ equal to $\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. Clearing denominators, we arrive at the expression $A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) = 6x^2 - 11x + 7$. Setting $x = 1$, we get $2A = 2$, so $A = 1$. Setting $x = 2$, we get $-B = 9$, so $B = -9$. Setting $x = 3$, we get $2C = 28$, so $C = 14$. Integrating results in the answer $\ln|x-1| - 9\ln|x-2| + 14\ln|x-3| + C$.