Math 217 (Meyer): Exam 1, Version 1

Your name Solutions

Circle your TA's name

Bryan Oakley Dima Kuzmenko

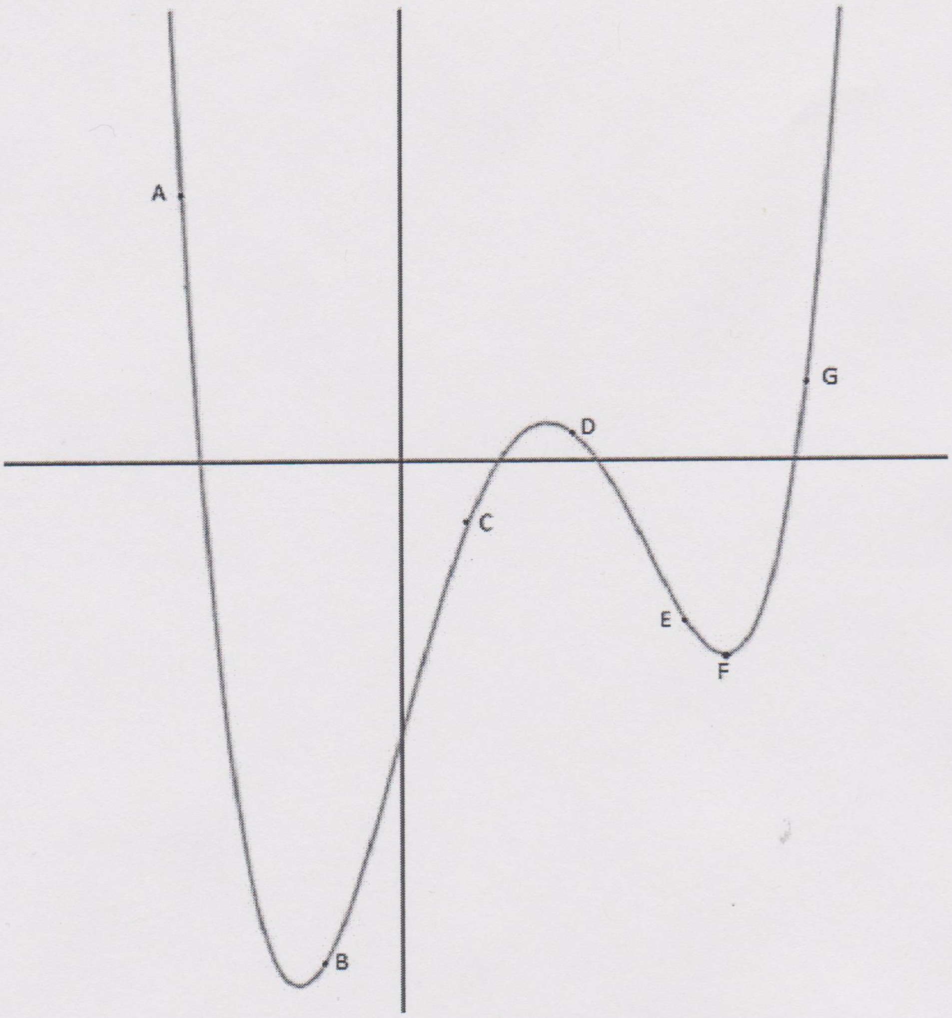
Bae Jun Park John Lynch

Show all your work in order to receive credit. A correct answer without any work will receive 0 credit. Partial credit will be given ONLY for work that is correct and relevant to the problem. Please, write your answers neatly.
All communications devices are turned OFF. No calculators.

Good luck!

Grade:

1.	/20
2.	/20
3.	/20
4.	/10
5.	/10
Total	/80



1. (7 + 6 + 7 points)

   (a) For the given points on the graph above,

   (i) find all the points with $f' > 0$

   B, C, G

   (ii) find all the points with $f'' > 0$

   A, B, E, F, G

(b) Find these limits

(i) $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}$$

$$\lim_{x\to 0}\frac{x-\sin x^{\nearrow 0}}{x\sin x_{\searrow 0}} \underset{\text{L'Hopital}}{=} \lim_{x\to 0}\frac{1-\cos x^{\nearrow 0}}{\sin x+x\cos x_{\searrow 0}} \underset{\text{L'Hopital}}{=} \lim_{x\to 0}\frac{\sin x^{\nearrow 0}}{\cos x+\cos x-x\sin x_{\to 2}} = 0$$

(ii) $\lim_{x\to 0}\left(\frac{\sin(x^2)}{1-\cos x}\right)$

L'Hopital

$$=\lim_{x\to 0}\frac{(\cos(x^2))\cdot 2x^{\nearrow 0}}{\sin x_{\searrow 0}}$$

$$=\lim_{x\to 0}\frac{2(\cos(x^2))-2x(\sin(x^2))\cdot 2x^{\nearrow 2}}{\cos x_{\searrow 1}}$$

$$=2$$

2. (20 points) For the function

$$f(x) = x + \frac{1}{x}$$

Find

(a) critical points, local maxima, minima, the intervals where f increases/decreases

$f(x) = \frac{x^2+1}{x}$ no x-intercepts, defined on $(-\infty, 0) \cup (0, \infty)$

$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = 0$ for $x = \pm 1$ ← critical points

$(-1, -2), (1, 2)$ ←

$x < -1$	$-1 < x < 0$	$0 \leq x < 1$	$x > 1$	
$+$	$-$	$-$	$+$	f'
↗	↘	↘	↗	

$(-1, -2)$ local max $\qquad$ $(1, 2)$ local min

(b) the intervals where f is concave up/concave down, has inflection points

$f''(x) = \frac{2}{x^3}$ no inflection pt.

$f''(x) < 0$ for $x < 0$ concave down on $(-\infty, 0)$

$f''(x) > 0$ for $x > 0$ concave up on $(0, \infty)$

(c) any asymptotes

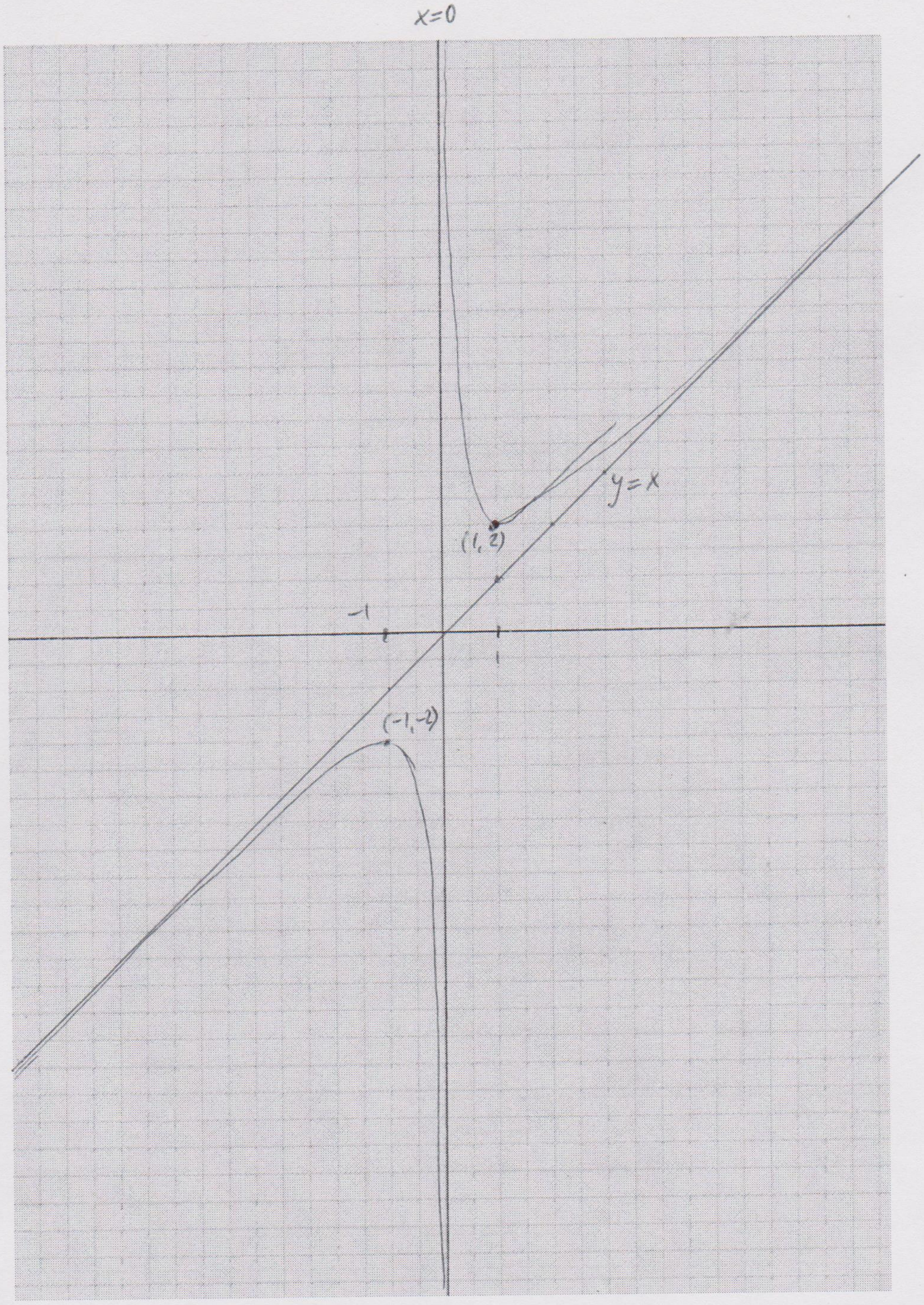
$(x^2+1) \div x = x$ $\Rightarrow$ f has asymptote $y = x$ as $x \to \infty$

$\cancel{x^2}$

1

vertical asymptote $x = 0$

(d) graph f


x=0
y=x
(1, 2)
-1
1
(-1, -2)

3. (20 points) For the parametric curve given by

$$x = 4\cos t, y = \sin t, 0 \leq t \leq 2\pi$$

Determine

(a) where the curve intersects the coordinate axes

$0 = x = 4\cos t$ for $t = \frac{\pi}{2}, \frac{3\pi}{2}$ then $(x,y) = (0,1), (0,-1)$

$0 = y = \sin t$ for $t = 0, \pi, 2\pi$, then $(x,y) = (4,0), (-4,0)$

(b) points with horizontal tangents, points with vertical tangents

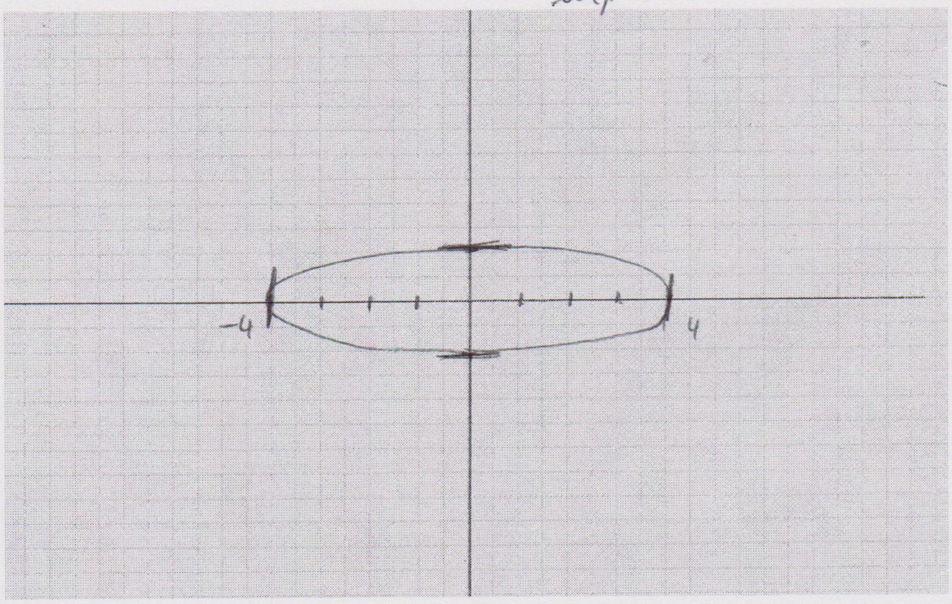
$x' = -4\sin t = 0$ for $t = 0, \pi, 2\pi$

$y' = \cos t = 0$ for $t = \frac{\pi}{2}, \frac{3\pi}{2}$

horizontal tangent when $y' = 0$, $x' \neq 0$ at $(0,1), (0,-1)$

vertical tangent when $y' \neq 0$, $x' = 0$ at $(4,0), (-4,0)$

ellipse



(c) where the curvature changes sign, if ever, and where the curvature is largest.

$$x'' = -4\cos t$$

$$y'' = -\sin t$$

$$K = \frac{x'y'' - y'x''}{\left[x'^2 + y'^2\right]^{3/2}} = \frac{(-4\sin t)(-\sin t) - \cos t(-4\cos t)}{(16\sin^2 t + \cos^2 t)^{3/2}}$$

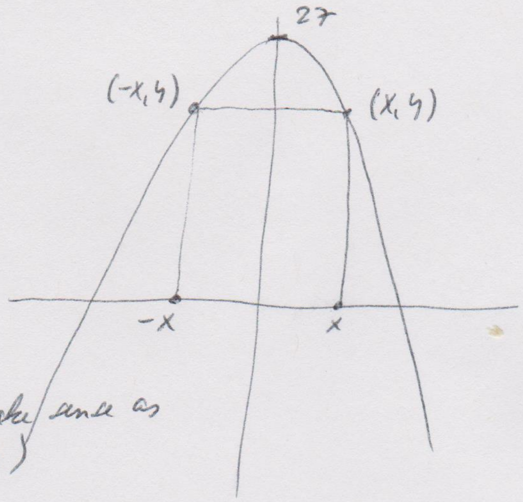
$$= \frac{4\sin^2 t + 4\cos^2 t}{(15\sin^2 t + 1)^{3/2}}$$

$$= \frac{4}{(15\sin^2 t + 1)^{3/2}} > 0 \quad \text{for all } t.$$

curvature never changes sign
and is largest at $t = 0, \pi, 2\pi$, i.e at $(4,0), (-4,0)$
(when denominator smallest)

(d) sketch the curve in the grid above

4. (10 points) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 27 - x^2$.
What is the largest area the rectangle can have? and what are its dimensions?



area of rectangle $= 2xy$; when $y = 27 - x^2$

area $A = 2xy = 2x(27 - x^2)$

$= 54x - 2x^3$

$A' = 54 - 6x^2 = 0$

$0 = 9 - x^2 \Rightarrow x = \pm 3$ (−3 does not make sense as a side length)

$A'' = -12x$

at $x = 3$ $A'' < 0$ so at $x = 3$ A has a local max.

$A(3) = 2 \cdot 3(27 - 9) = 6 \cdot 18 = 108$

dimensions: 6×18

width height

5. (4 + 6 points)

(a) Solve the following equation: $\log_7(x+2) + \log_7(x-4) = 1$

$\Rightarrow \log_7[(x+2)(x-4)] = 1$

$\Rightarrow$

$$
\begin{aligned}
(x+2)(x-4) &= 7^1 = 7 \\
x^2 + 2x - 4x - 8 &= 7 \\
x^2 - 2x - 15 &= 0 \\
(x+3)(x-5) &= 0
\end{aligned}
$$

$\Rightarrow x = -3, 5$

but $\log_7(-3+2) = \log_7(-1)$ is not defined.

$5+2>0,\ 5-4>0 \quad \Rightarrow$ the only solution is $x=5$.

(b) The half life of radon-222 is about 92 hours.
About how many hours will it take for a sample of radon-222 to decay to $\frac{1}{1000}$ of the original amount?

after $k \cdot 92$ hours	amount
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$
5	$\frac{1}{32}$
6	$\frac{1}{64}$
7	$\frac{1}{128}$
8	$\frac{1}{256}$
9	$\frac{1}{512}$
10	$\frac{1}{1024}$

after $10 \cdot 92$ hours, there is about $\frac{1}{1000}$ of the original amount left $= 920$ (or $9 \cdot 92 = 828$ hours to achieve $\approx \frac{1}{500}$ of original amount)

Alternatively:

$$A(t) = A(0)\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$\frac{1}{1000}\cancel{A(0)} = \cancel{A(0)} \cdot \left(\frac{1}{2}\right)^{\frac{t}{92}}$$

$$\ln\frac{1}{1000} = \frac{t}{92}\ln\left(\frac{1}{2}\right)$$

$$\left(\frac{\ln\frac{1}{1000}}{\ln\left(\frac{1}{2}\right)}\right) \cdot 92 = t$$

$$\left(\frac{-\ln 1000}{-\ln 2}\right) \cdot 92 = t$$

$$\boxed{\left(\frac{\ln 1000}{\ln 2}\right) \cdot 92 = t}$$

Math 217 (Meyer): Exam 1, Version 2

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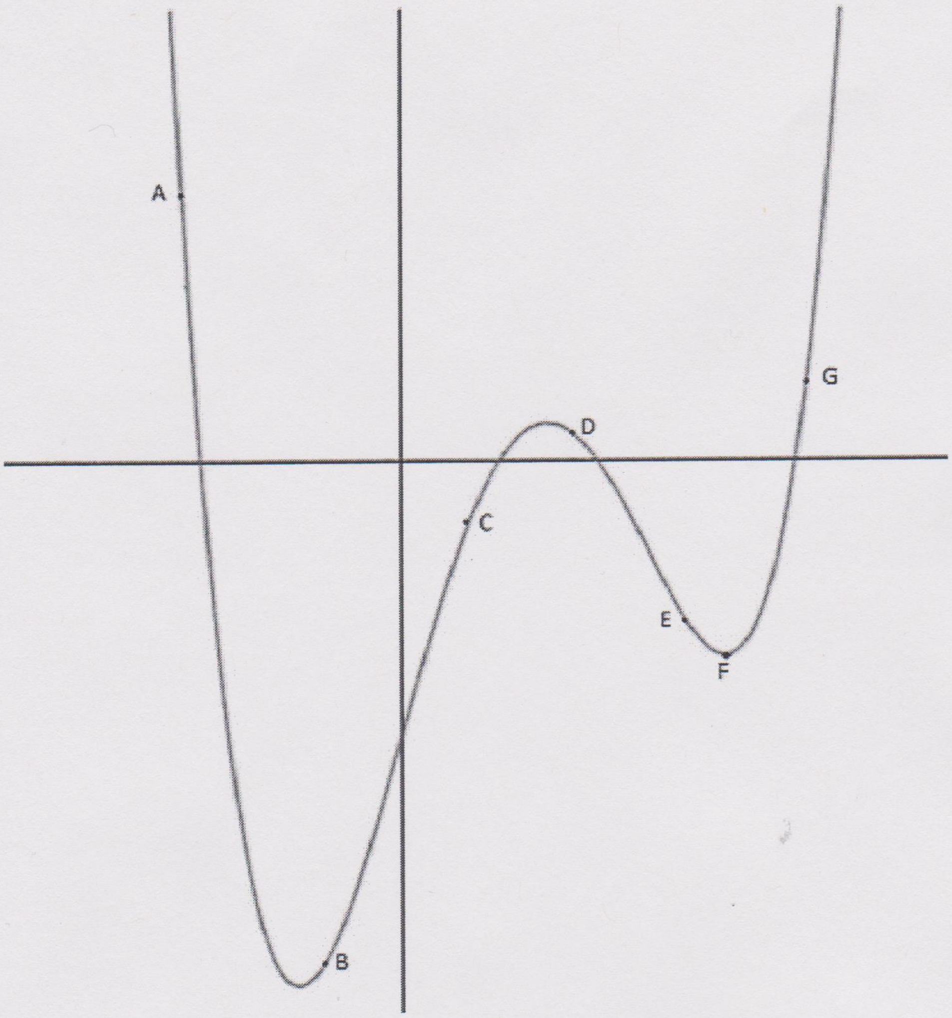


1. (7 + 6 + 7 points)

(a) For the given points on the graph above,

(i) find all the points with $f' > 0$

B, C, G

(ii) find all the points with $f'' > 0$

A, B, E, F, G

(b) Find these limits

(i) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$$

$$\lim_{x\to 0} \frac{\sin x - x}{x \sin x} \begin{matrix}\nearrow 0\\ \searrow 0\end{matrix} \underset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} \begin{matrix}\nearrow 0\\ \searrow 0\end{matrix}$$

$$\underset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x} \begin{matrix}\nearrow 0\\ \searrow 2\end{matrix}$$

$$= 0$$

(ii) $\lim_{x\to 0}\left(\frac{\sin(x^2)}{\cos x - 1}\right)$ $\begin{matrix}\nearrow 0\\ \searrow 0\end{matrix}$

$$\underset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{\cos(x^2)\cdot 2x}{-\sin x} \begin{matrix}\nearrow 0\\ \searrow 0\end{matrix}$$

$$= \lim_{x\to 0} \frac{2\cos(x^2) + 2x(-\sin(x^2))\cdot 2x}{-\cos x} \begin{matrix}\nearrow 2\\ \searrow -1\end{matrix}$$

$$= -2$$

2. (20 points) For the function

$$f(x) = x + \frac{4}{x}$$

Find

(a) critical points, local maxima, minima, the intervals where f increases/decreases

f defined on $(-\infty, 0) \cup (0, \infty)$, no x-intercepts

$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = 0$ for $x = \pm 2$ ← critical points.

$(-2, -4)\ (2, 4)$

	$x<-2$	$-2<x<0$	$0<x<2$	$x>2$
f'	+	−	−	+
	↗	↘	↘	↗

$(-2, -4)$ local max

$(2, 4)$ local min

(b) the intervals where f is concave up/concave down, has inflection points

$f''(x) = \frac{8}{x^3}$ ⇒ no inflection pts.

$f''(x) < 0$ for $x < 0$ concave down. $(-\infty, 0)$

$f''(x) > 0$ for $x > 0$ concave up $(0, \infty)$

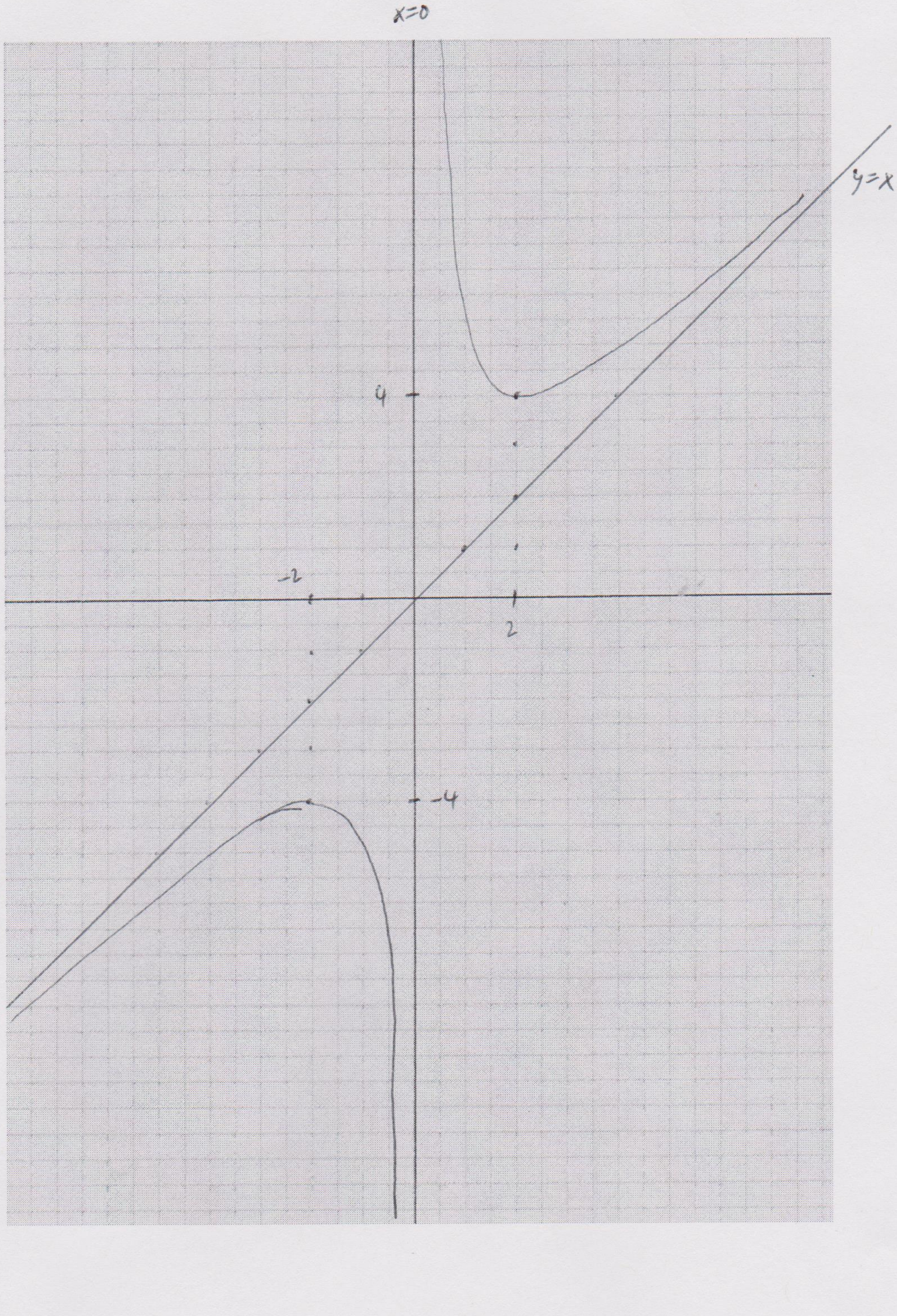
(c) any asymptotes

vertical asymptote $\underline{x = 0}$ because $f(x) = \frac{x^2+4}{x}$

slant asymptote: $(x^2+4) \div x = x + \frac{4}{x} \to 0$ as $x \to \infty$

$\underline{y = x}$

(d) graph f


x=0
y=x
4
-4
-2
2

3. (20 points) For the parametric curve given by

$$x = 3\cos t, y = \sin t, 0 \leq t \leq 2\pi$$

Determine

(a) where the curve intersects the coordinate axes

$0 = x = 3\cos t$ for $t = \frac{\pi}{2}, \frac{3\pi}{2}$ then $(x,y) = (1,0), (0,-1)$

$0 = y = \sin t$ for $t = 0, \pi, 2\pi$ then $(x,y) = (3,0), (-3,0)$

(b) points with horizontal tangents, points with vertical tangents

$x' = -3\sin t = 0$ for $t = 0, \pi, 2\pi$

$y' = \cos t = 0$ for $t = \frac{\pi}{2}, \frac{3\pi}{2}$

horizontal tangent when $y' = 0$ and $x' \neq 0$ then $(1,0), (0,-1)$

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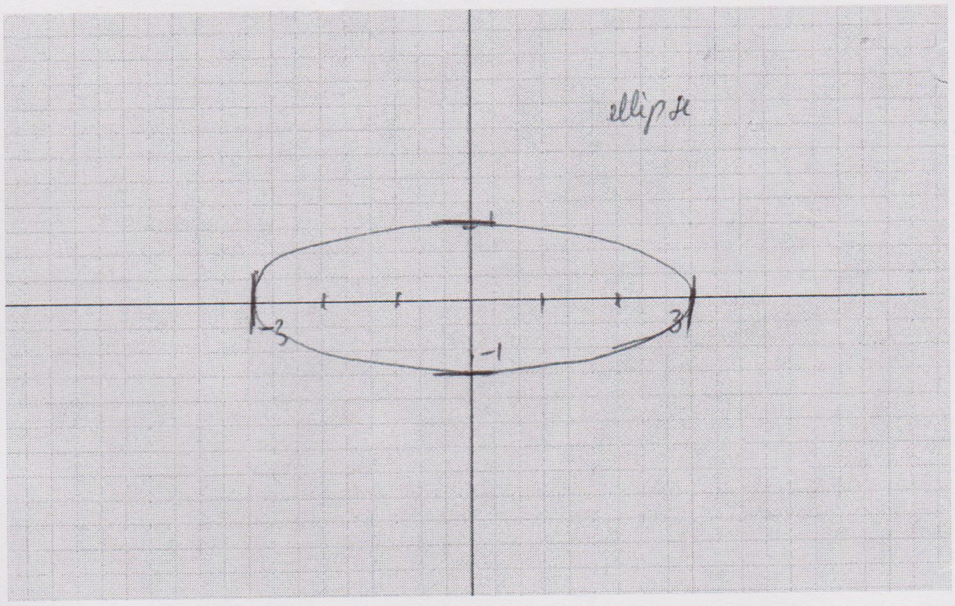


(c) where the curvature changes sign, if ever, and where the curvature is largest.

$$K = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}} \qquad x'' = -3\cos t \qquad y'' = -\sin t$$

$$= \frac{(-3\sin t)(-\sin t) - (-3\cos t)(\cos t)}{(9\sin^2 t + \cos^2 t)^{3/2}} =$$

$$= \frac{3\sin^2 t + 3\cos^2 t}{(8\sin^2 t + 1)^{3/2}}$$

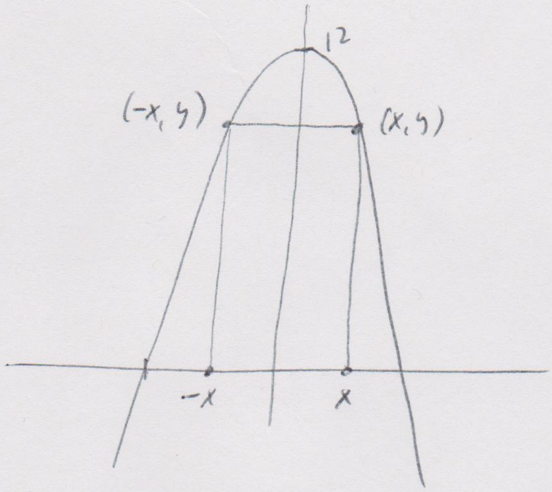
$$= \frac{3}{(8\sin^2 t + 1)^{3/2}} \neq 0 \quad \text{for any } t \quad \text{actually it's} > 0 \text{ for all } t.$$

curvature never changes sign

curvature is largest, when denominator smallest (at $t=0$), then curvature $= 3$

(d) sketch the curve in the grid above

4. (10 points) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$.
What is the largest area the rectangle can have? and what are its dimensions?



area of the rectangle $= A = 2xy$

where $y = 12 - x^2$

so $A = 2x(12 - x^2) = 24x - 2x^3$

$A' = 24 - 6x^2$

$A' = 0 = 4 - x^2 \Rightarrow x = \pm 2$

-2 doesn't make sense as a side length

$\Rightarrow x = 2$

$A'' = -12x$

$A''(2) = -24 < 0$ so at $x = 2$ A has a local max

$A(2) = 2 \cdot 2 \cdot (12 - 4) = \underset{\text{width}}{4} \cdot \underset{\text{height}}{8} = \underset{\text{maximal area}}{32}$

5. (4 + 6 points)

(a) Solve the following equation: $\log_7(x+2) + \log_7(x-4) = 1$

$$\Rightarrow \log_7[(x+2)(x-4)] = 1$$

$$\Rightarrow (x+2)(x-4) = 7^1 = 7$$

$$x^2 + 2x - 4x - 8 = 7$$

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