The differential equation $y' = y$ and the function $\exp(x)$

Motivation: Compound Interest [TF 6-11]
Suppose you deposit an amount A_0 in a bank account paying an interest r (*e.g.* $r = 6\%$ *per year*) with a compounding period of T, then the amount after one period will have grown to $A(T) = (1+rT)A_0$. After the next compounding period it will be $A(2T) = (1+rT)A(T) = (1+rT)^2 A_0$ and so on. After n periods it will be

$$A(nT) = (1+rT)^n A_0. \tag{1}$$

The amount in deposit increases by a factor $(1+rT)$ after each period: $A(t+T) = (1+rT)A(t)$. We can write this in the form of a *difference equation:*

$$\frac{A(t+T)-A(t)}{T} = rA(t). \tag{2}$$

The solution of this difference equation is (1), check it! It should be clear that the compounding period is as important as the interest rate itself. Banks usually quote the interest rate (in units of yr^{-1}) and the *Annual Percentage Yield (APY)* instead of the compounding period. What is the connection between r, T and the APY? If the interest rate is 6% and T is one month, what is the APY? If $r = 6\%$ and $T = 1$ *week*, what is the APY?

• *Continuous compounding:* $T \to 0$
Continuous compounding corresponds to the limit $T \to 0$, then the solution (1) becomes $A(0) = A_0$. This is just the initial amount, what about $A(t > 0)$?! The difference equation (2) does not become so useless in the limit $T \to 0$. The difference quotient becomes a derivative and the equation becomes the *differential equation*

$$\frac{dA}{dt} = rA. \tag{3}$$

We need to find the solution of this equation to determine $A(t)$ and the proper limit of (1).

• *Non-dimensionalization*
Before trying to find the solution of (3) it is useful to reduce it to its bare essentials by using *non-dimensional variables.* This is an important step in mathematical modeling of scientific and engineering problems. The variables entering the problem are: A measured in dollars or cents, r an inverse time and t time. The units of r and t must be compatible, if r is taken per year, then t should be measured in years.
Let $x = rt$ and $y(x) = A(t)/A_0$ (assuming the original amount A_0 is not zero) then x and y are non-dimensional. By the chain rule and equation (3):

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{rA_0}\frac{dA}{dt} = y.$$

So the differential equation that we would like to solve is

$$y' = y \tag{4}$$

with the *initial condition* $y(0) = 1$. This is the equation for which separation of variables [TF 4-2] $y' = y \Leftrightarrow y^{-1}dy = dx$ breaks down as $\int y^n dy = y^{n+1}/(n+1)$ but here $n = -1$.

• *Solution*
Sketch solution using *piecewise linear* approximation over intervals of size Δx *(i.e. Euler's method)*. Algebraically, Euler's method gives $y(n\Delta x) \approx (1+\Delta x)^n y(0)$ (*cf.* (1) and Fig. 1).
Solution by polynomial approximation (*i.e.* power series, *i.e.* Taylor's formula [TF 3-10]):

$$y(x) \approx y(0) + y'(0)\, x + \frac{1}{2} y''(0)\, x^2 + \frac{1}{3!} y'''(0)\, x^3 + \cdots \tag{5}$$

The differential equation $y' = y$ implies $y = y' = y'' = \cdots$, and $y(0) = 1$ thus $y(0) = y'(0) = y''(0) = y'''(0) = \cdots = 1$ and

$$y(x) \approx 1 + x + x^2/2 + x^3/6 + x^4/4! + x^5/5! + \cdots \tag{6}$$

See Fig. 2. You can check using term-by-term differentiation that this is indeed the solution of $y' = y$, *if the sum does not end!*. One can make sense of these infinite sums using limits [TF 16].
The solution (6) is not a simple power law or polynomial (the sum does not end). It is a new function defined by the differential equation $y' = y$ with $y(0) = 1$.
But is the solution to that equation unique? After all $y' = \sqrt{y}$ with $y(0) = 0$ has *two* solutions $y(x) = 0$ and $y(x) = x^2/4$ (check it).

• *Uniqueness:*
(1) by variation of parameters. Let $y_1(x)$ be a solution of $y' = y$ with $y(0) = 1$. Let $y(x) = u(x) y_1(x)$ be another solution then $u' y_1 = 0$ and $u = 1$, so $y(x)$ is identical to $y_1(x)$. [Tricky point: what if $y_1(x) = 0$?]
(2) another proof: Assume $u(x)$ and $v(x)$ are two different solutions. Consider $w(x) = u(x)v(a-x)$ where a is any constant. Then by the product and chain rules $w' = 0 \quad \forall a, x$ hence $u(x)v(a-x) = v(a)$. Of course $v(x)v(a-x) = v(a)$ by the same reasoning. Hence $u(x)v(a-x) = v(x)v(a-x)$ for all a, x and $u(x) = v(x)$.
Either way, the solution of $y' = y$ with $y(0) = 1$ is indeed unique, let's call it $y = \exp(x)$.

• *Main property of* $\exp(x)$*:*
Note that $\exp(\alpha x)$ is the unique solution of $y' = \alpha y$ with $y(0) = 1$, however

$$\frac{d}{dx}[\exp(x)]^\alpha = \alpha[\exp(x)]^{\alpha-1}[\exp(x)]' = \alpha[\exp(x)]^\alpha$$

so $[\exp(x)]^\alpha$ also satisfies $y' = \alpha y$ with $y(0) = 1$, hence, by uniqueness,

$$\exp(\alpha x) = [\exp(x)]^\alpha.$$

Nice application of the uniqueness proof. In particular, $\exp(x) = [\exp(1)]^x \equiv e^x$ where

$$e = 1 + 1 + 1/2 + 1/3! + 1/4! + 1/5! + 1/6! + \ldots = 2.718...$$

Note that Euler's method gives $y(t) \approx (1 + t/n)^n$ and this suggests $\lim_{n\to\infty}(1 + t/n)^n = e^t$.

The differential equation $y'' + y = 0$ can be treated similarly as well as in 4-4 (see 18-12 for a physical introduction). One could even go on to e^{ix}...

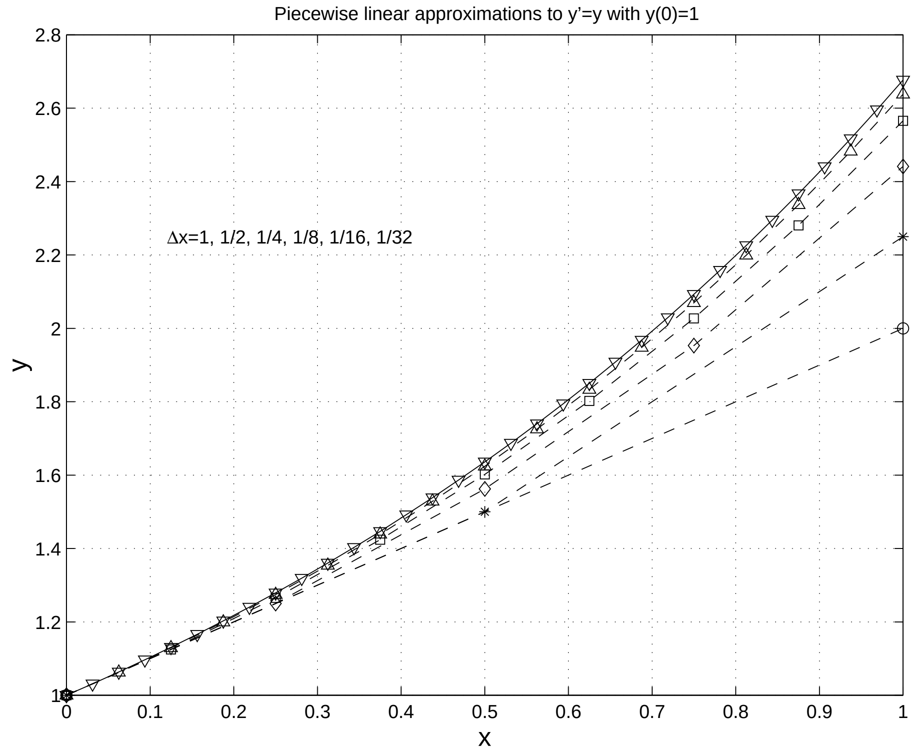

Figure 1: Piecewise linear approximations (Euler's method)

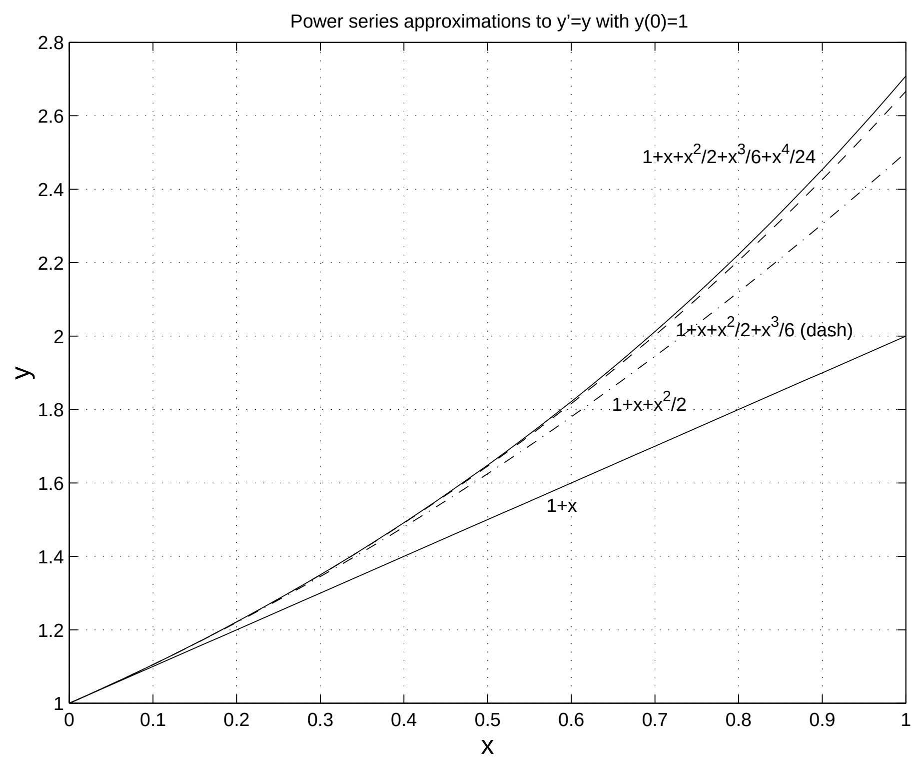

Figure 2: Polynomial approximations (Taylor series)