

Calculus 222 Instructor's guide
F. Waleffe, Spring 2001

TEXTBOOK: *Calculus and Analytic Geometry* by Thomas and Finney, 5th Ed.

Calculus 222 is a challenging course but it contains many classical calculus topics and can be fun to teach. For the students, it has the reputation of being the most difficult course in the Calculus sequence. The difficulty comes from the fact that it typically contains an eclectic collection of topics in between single variable and multi-variable calculus. Differential equations were added a few years ago at the request of the School of Engineering. Most engineering students must take the calculus sequence but are not required to take 319, although differential equations are important in engineering.

I. Techniques of Integration: Substitution, integration by parts, partial fractions, Maple.

- **7.2,3,4** are really about substitution but involve a lot of trigonometry. Emphasize those two distinct aspects. It is a good idea to have an early quiz (*e.g.* first discussion) on sections 2-9,10 to refresh the trigonometry so it does not get in the way of substitution. Example 5 in 7-2 is better done by partial fractions in week 2 than with the preposterous trick on p. 340. Example 6 is a nice candidate for required reading and some exam question. The last part of 7-4 (about changing limits of integration in definite integrals) is very important.
- **7-6** Do not forget $\int dx/\cos x$ as an application of substitution and partial fractions.
- Week 3 is open. This is a good time to introduce students to **Maple**. Inform your TAs that they are responsible for this elementary Maple. If you cannot demonstrate Maple in lectures, you or your TAs maybe able to take students to B107, the computer classroom. The students can also go to VV 101 (on top of the library). Some examples of things to do in Maple are given at the end of this guide.
- Formulas (4)-(6) on p. 370, Sect. 7-8 should be covered but not necessarily in the lectures.

II. Improper integrals and Series

- **7-10, Improper Integrals** is an important section. It may be useful to generalize the fundamental theorem as $\int_a^b (dF/dx)dx = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$.
- **Series.** Improper integrals (and Riemann Sums) provide a good introduction to Series. Sequences and Series are covered in Chapter 16, but do not follow the book closely. There are too many topics there that will detract from the essential material and become a time sink. The objective in this part of the course is to give a basic discussion of Taylor series and its applications. There is no need, nor time, to cover sequences in detail. Discuss algebraic series $\sum_{n=1}^{\infty} n^p$ and geometric series $\sum_{n=1}^{\infty} q^n$ by comparison to the integrals $\int_1^{\infty} x^p dx$ and $\int_1^{\infty} q^x dx$, respectively, using Riemann sums. The connection between the discrete and continuous algebraic sums is excellent. Note that the relative change in two consecutive terms are

$$\frac{(n+1)^p - n^p}{n^p} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{while} \quad \frac{q^{n+1} - q^n}{q^n} = q - 1, \quad \forall n$$

(so the discrete function gets relatively closer to the continuous one in the algebraic case but not in the geometric case). Although the geometric series is not well approximated by an exponential integral, the latter still suggests that the series will converge if $0 < q < 1$. Discuss the geometric series (Section 16-4), then the integral, comparison and ratio tests in 16-5. A few remarks about absolute convergence and alternating series would be good but there is certainly no need for Leibniz's theorem (16-7).

- **16-8,9,10,13.** Taylor Series. Important material for applications. Students should be comfortable with finding the Taylor series of a function.

III. Differential Equations

The objectives here are to introduce the concepts of differential equations, constants of integration, initial and/or boundary conditions (18-1,2,12). Limit yourself to separable equations (18-3), linear

1st order (18-5) and linear, homogeneous, 2nd order (18-9). The latter are most important for engineering applications (damped oscillators, RLC circuits). Solution of differential equations by Taylor Series is short-changed in our curriculum (not covered even in 319). This is unfortunate as many engineering and science students will encounter many special functions (Bessel, Legendre, Hermite,...) in their career and the concept of expanding an unknown function in terms of known functions is fundamental to numerical methods. Solving $y' = y$, $y'' = \pm y$ and $y'' = xy$ by Taylor series are good exercises.

IV. Coordinates, Curves, Vectors

- **10-1,2,3,5.** Polar Coordinates and Area. A nice Maple exercise in the same vein as the week 3 example below is to compute the area of the “rounded-square” (or “squared-circle”) $x^4 + y^4 = 1$. The trapezoidal rule in x converges painfully, but the trapezoidal rule in θ using polar coordinates works like a charm.
- **11-1,2,3,4,5** Vectors and parametric equations. There is a lot of nice, classic calculus material here. Maple is again very useful here (*e.g.* in 11-3 calculating time to slide down various “ski slopes” are good exercises to assign).
- **11-6,7,8,9** Products of vectors, lines and planes. Many students will have already seen this material in physics and mechanics courses.
- **12-2,3,4,5,6** Vector functions and their derivatives. Lots of nice Calculus material again. Skip binormal in 12-4, examples 1 and 4 as well as formulas (14-16) in 12-5. Vectors in polar coordinates is important. Planetary motion gives a nice ending to the course and a good illustration of the material.

There is a nice web site about “famous curves,” the address is
<http://www-history.mcs.st-andrews.ac.uk/history/Java/index.html>

MAPLE example material for WEEK 3:

Here is some information about starting Maple, trying some integrals and demonstrating what a little substitution can do, even if the integral cannot be done analytically, provided one can change the limits of integrations.

Start Maple, click on the “help” button in the top right corner of the Maple window and select the “New user’s tour”. Take a look at “Calculus”. Try out a few integrals, for instance

```
int(1/cos(x),x);            $\int dx/\cos(x)$ 
int((1-x^2)^(1/2),x);       $\int \sqrt{1-x^2}dx$ 
int(cos(x)*(1-x^2)^(1/2),x);  $\int \cos(x)\sqrt{1-x^2}dx$ 
```

(the semi-colon ; is required syntax). Maple will not be able to do the last one (you won’t either) and will return the input in “pretty form”. Next, try definite integration:

```
int(cos(x)*(1-x^2)^(1/2),x=0..1);  $\int_0^1 \cos(x)\sqrt{1-x^2}dx$ 
```

this should give you $\pi/2$ BesselJ(1,1) which won’t make any sense to the students. You may want to select the “help” button again to do a “topic search” on BesselJ. The answer still won’t make sense to the students but they will see a differential equation that many will see again in later courses.

You can plot functions with the command

```
plot(sqrt(1-x^2),x=0..1);
```

or for multiple curves on the same plot using a list of functions [.,.]

```
plot( [sqrt(1-x^2), cos(x)*sqrt(1-x^2)], x=0..1);
```

Call the “student” package: `with(student)`; Try repeated numerical integration of this integral using the trapezoidal rule for instance.

```
f:=x -> cos(x)*(1-x^2)^(1/2);      # defines the function f(x)
evalf(trapezoid(f(x),x=0..1,2);      .6300043962
evalf(trapezoid(f(x),x=0..1,4);      .6705302268
evalf(trapezoid(f(x),x=0..1,8);      .6840649145
evalf(trapezoid(f(x),x=0..1,16);     .6887226369
evalf(trapezoid(f(x),x=0..1,32);     .6903479050
```

`trapezoid` is the trapezoidal rule, the last argument is the number of subintervals (see “topic search” for further details). `evalf` evaluates in floats (*i.e.* numerically). Note the very slow numerical convergence. Now try the substitution $x = \sin \theta$ as in example 6 of 7-4. The integrand is now $f(\sin \theta) \cos \theta$. Change the limits of integrations and try the trapezoidal rule again:

```
evalf(trapezoid(f(sin(a))*cos(a),a=0..Pi/2,2); .6912464369
evalf(trapezoid(f(sin(a))*cos(a),a=0..Pi/2,4); .6912298438
evalf(trapezoid(f(sin(a))*cos(a),a=0..Pi/2,8); .6912298438
```

Note how much faster the trapezoidal rule converges. Trapezoidal rule on the “raw” form converged slowly (slower than the quadratic convergence advertized on p. 212 because of the singular derivatives at $x = 1$) but the same trapezoidal rule converges exponentially in the transformed variable, provided one knows how to change the limits of integration!