

Problem 3 of PS4 asked for the moment of inertia around the y -axis, I_y say, of the flattened circle $x^4 + y^4 = 1$ (take $R = 1$, for other R multiply the result for unit size by R^3).

$$I_y = \int_{\mathcal{C}} x^2 ds = \int_{\mathcal{C}} x^2 \sqrt{dx^2 + dy^2}$$

where ds is the line element and $\mathcal{C}$ denotes the total curve. By Pythagoras, $ds^2 = dx^2 + dy^2$, because the x and y directions are orthogonal.

By symmetry we only need to compute over a quarter of the rim, *i.e.* $x, y \geq 0$. Let's call that the curve $\mathcal{C}_+$, so $I_y = 4 \int_{\mathcal{C}_+} x^2 ds$.

What variable will we take as our independent variable? x ? y ? or something else?

$$\int_{\mathcal{C}_+} x^2 ds = \int_0^1 x^2 \sqrt{1 + (dy/dx)^2} dx = \int_0^1 x^2 \sqrt{(dx/dy)^2 + 1} dy$$

Mathematically, these are equivalent, but computationally, they are not. For instance if we take x as the independent variable then $y = (1 - x^4)^{1/4}$ and $dy/dx = -x^3(1 - x^4)^{-3/4}$. This leads to a singularity near $x = 1$ of the form $(1 - x)^{-3/4}$ (because $1 - x^4 = (1 - x)(1 + x)(1 + x^2)$) and it's only the $(1 - x)$ that leads to the singularity). As a result, the trapezoidal rule or Simpson's rule will break their noses because the integrand is singular at the end point $x = 1$. Taking y as the independent variable also leads to a singularity at $y = 1$ coming from $dx/dy = -y^3(1 - y^4)^{-3/4}$, but a weaker one because of the $x^2 = (1 - y^4)^{1/2}$ factor in the integrand. Now the singularity at $y = 1$ is "only" of the form $(1 - y)^{-1/4}$, better, but still not good enough for the trapezoidal rule. The change of variables $y = \cos a$ would get rid of the singularity because $(1 - y^2)^{-1/4} = (\sin a)^{-1/2}$ but $dy = -\sin a da$ and this extra factor of $\sin a$ kills out the singularity. Now the trapezoidal rule would work, however it would not converge as well as the usual quadratic behavior because of the remaining $\sqrt{\sin a}$. We can do better.

Let's go back to the nastier x formulation. The singularity is of the form $(1 - x)^{-3/4}$ and this comes directly from $d/dx (1 - x)^{1/4}$. So the change of variable $1 - x = u^4$ gets rid of that annoying singularity. Indeed, $u = (1 - x)^{1/4}$ and

$$\frac{dx}{(1 - x)^{3/4}} = -4du$$

$$\int_0^1 x^2 \sqrt{1 + (dy/dx)^2} dx = \int_0^1 4x^2 \sqrt{1 + (dy/dx)^2} (1 - x)^{3/4} du$$

where $x = 1 - u^4$ in that second integral ($\int du$). Now the trapezoidal rule works like it's supposed to:

```
>> N=10; u=[0:1/N:1]; x=(1-u.^4);
>> dsdu=sqrt(x.^6 + (1-x.^4).^(3/2))./((1+x.^2).*(1+x)).^(3/4);
>> fu=4*x.^2.*dsdu; trapz(u,fu)
ans =
    1.0433
```

Simpson's rule would work also. But we can do even better if we use polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, then from $x^4 + y^4 = 1$, we get $r = (\cos^4 \theta + \sin^4 \theta)^{-1/4}$. This looks ugly but note that r can never be zero, so we won't have singularities! In polar coordinates the integral becomes

$$\int_{c_+} x^2 ds = \int_0^{\pi/2} x^2 \sqrt{dr^2 + r^2 d\theta^2}$$

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>> N=10; t=[0:1/N:1]*pi/2; r=(cos(t).^4+ sin(t).^4).^(-1/4); x=r.*cos(t);
>> drdt=0.25*sin(4*t).*r.^5; ft=x.^2.*sqrt(drdt.^2 + r.^2);
>> trapz(t,ft)
ans =
    1.0436
```

Here are the results of both approaches, u and θ , for various N :

N	u-formulation	polar formulation
10	1.04332793260515	1.04363903673463
20	1.04362102571569	1.04364092060518
40	1.04363967224164	1.04364092061353
80	1.04364084251396	1.04364092061353

Clearly the polar coordinates idea is much better. The answer is fully converged at $N=40$ (well before $N=40$ in fact), while the trapezoidal rule for $\int_0^1 f(u)du$ is still working on the 6th decimal. The trapezoidal rule for the "u"-method works as expected, error of $O(N^{-2})$ but for the θ -method (polar coords) we get *exponential convergence*. That's fabulous!

So this was quite an interesting exercise. The "right" change of variables was $x = r \cos \theta$ with $y = r \sin \theta$ and then $r = (\cos^4 \theta + \sin^4 \theta)^{-1/4}$ for $x^4 + y^4 = 1$, instead of the seemingly close $x = \cos \theta$ with $y = (1 - \cos^4 \theta)^{1/4}$ for $x^4 + y^4 = 1$.