Problems due Thursday 10/6-13/2016. No email or web submissions.

You can discuss the homework with other students. You must cite your collaborators.
Read `http://www.math.wisc.edu/~waleffe/GFD2011lectures/GFD1.pdf`

1. Drazin # 2.8 (ii) but generalized using $\lambda_1 < 0$ instead of -2ϵ, and $\lambda_2 < 0$ instead of $-\epsilon$. In applications $|\lambda_1| \leq |\lambda_2| \ll 1$ typically. Show that the matrix is defective (only one eigenvector) when $\lambda_1 = \lambda_2 = \lambda$ and find the general solution in that case, showing that it is a combination of $\exp(\lambda t)$ and $t\exp(\lambda t)$ terms. The latter behavior, $t\exp(\lambda t)$ is known as 'transient (algebraic) growth'. For the general case $\lambda_1 \neq \lambda_2$, find an explicit form for the transient growth (aka the 'forced response') $R(t)$, say, the transiently growing component of the system corresponding to initial condition $x_0 = 0$, $y_0 = 1$. Show that $R(t)$ is bracketed by the transient growth in the defective cases with λ_1 as the repeated eigenvalue, and with λ_2 as the repeated eigenvalues:

$$te^{\lambda_2 t} \leq R(t) \leq te^{\lambda_1 t}.$$

2. (Drazin 8.25. Note that Drazin uses x for the main flow direction and z for the shear direction as is common in Atmosphere and Ocean Sciences. The engineering literature typically uses x for the flow direction and y for the shear direction, as below).

   Consider the unbounded shear flow $\boldsymbol{U} = S_0 y\,\hat{\boldsymbol{x}}$ + a 'small' perturbation $\boldsymbol{u}(\boldsymbol{r}, 0) = \Omega\hat{\boldsymbol{x}} \times \boldsymbol{r}$ where $\boldsymbol{r}$ is the usual position vector, with $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + x\hat{\boldsymbol{z}}$ in cartesian form.

   - What does 'small' mean in this context where the base flow and the perturbation both go to infinity as $|\boldsymbol{r}| \to \infty$?
   - Show that the linearized Navier-Stokes equations for $\boldsymbol{u}(\boldsymbol{r}, t)$ yield

   $$\boldsymbol{u} = \Omega S_0\, t\,\hat{\boldsymbol{x}} + \Omega\,\hat{\boldsymbol{x}} \times \boldsymbol{r} \tag{1}$$

     that is growing algebraically for all times and illustrates the 'lift up' effect.
   - Show that the full Navier-Stokes equations for $\boldsymbol{v} = \boldsymbol{U} + \boldsymbol{u}$ admit a solution of the form

   $$\boldsymbol{v} = (S(t)y + A(t)z)\;\hat{\boldsymbol{x}} + \Omega\,\hat{\boldsymbol{x}} \times \boldsymbol{r} \tag{2}$$

     with Ω constant. Find and solve the equations for $S(t)$ and $A(t)$. Compare and reconcile these results with the linear solution.

3. (Drazin 2.19, Craik and Criminale 1986)

   Consider velocity fields of the form $\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{A}(t) \cdot \boldsymbol{r}$ where $\boldsymbol{A}(t)$ is a uniform tensor (matrix), independent of position $\boldsymbol{r}$. Thus $v_i = A_{ij}x_j$ in index notation with A_{ij} (possibly) time dependent. What are the conditions on $\boldsymbol{A}(t)$ for $\boldsymbol{v}$ to be a solution of the full Navier-Stokes equations (in an unbounded domain)?

4. We found perturbation solutions of the form $\boldsymbol{u}(t)e^{i\boldsymbol{k}(t)\cdot\boldsymbol{r}}$ for the unbounded base shear flow $\boldsymbol{U} = Sy\hat{\boldsymbol{x}}$ with $\boldsymbol{k}(t) = \boldsymbol{k}(0) - (\hat{\boldsymbol{x}}\cdot\boldsymbol{k}(0))\,St\,\hat{\boldsymbol{y}}$ and found two distinct types of transient growth: the Orr mechanism (aka 'Venitian blind' effect) and the 'lift up' effect. We also found that the equation for the $\hat{\boldsymbol{y}}$ velocity component decoupled from the $\hat{\boldsymbol{x}}$ component after elimination of the pressure.

   Consider 2D perturbations $\boldsymbol{u} = \big(u(x,y,t), v(x,y,t)\big)$ to the base shear flow $\boldsymbol{U} = Sy\hat{\boldsymbol{x}}$ for incompressible, *inviscid* flow.

   - Derive the linearized equations for u and v.
   - Eliminate the pressure and the streamwise velocity u to derive the v equation

     $$(\partial_t + Sy\partial_x)\nabla^2 v = 0 \tag{3}$$

     where $\nabla^2 = \partial_x^2 + \partial_y^2$.
   - Verify that

     $$v = A\frac{\alpha^2+\beta^2}{\alpha^2+(\beta-\alpha St)^2}\,e^{i(\alpha x+(\beta-\alpha St)y)} + c.c. \tag{4}$$

     is a solution of the v equation, for any real α, β and amplitude A, where *c.c.* means 'complex conjugate'. This solution is a Kelvin mode solution.
   - Assume now that there are two parallel walls at $y = \pm h$ with boundary conditions $v = 0$ at those walls. Find a solution of (3) that satisfies the boundary conditions $v = 0$ at $y = \pm h$ in the form

     $$v = v_K + v_P$$

     where v_K is (4) and v_P is a solution of Laplace's equation $\nabla^2 v_P = 0$. What are the boundary conditions for v_P?
   - Do the walls enhance or reduce the transient growth?
   - A few illustrative plots of $v(x,y,t)$ and/or $\boldsymbol{u}(x,y,t)$ or the instantaneous streamfunction $\psi(x,y,t)$ such that $u = \partial_y\psi$, $v = -\partial_x\psi$, would be good.