F. Waleffe, Math 705 lecture of 09/09/2002

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The time rate of change of $\int_{V(t)} f(\mathbf{X},t)\,d\mathbf{X}$, where $V(t)$ is a material volume, i.e. a volume that moves with the fluid, is

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} f(\mathbf{X},t)\,d\mathbf{X} &= \frac{d}{dt}\int_{V_0} f(\mathbf{X}(\boldsymbol{\alpha},t),t)J(\boldsymbol{\alpha},t)\,d\boldsymbol{\alpha} \\ &= \int_{V_0}(\frac{Df}{Dt}J + f\frac{DJ}{Dt})\,d\boldsymbol{\alpha} \\ &= \int_{V(t)}\frac{Df}{Dt}d\mathbf{X} + \int_{V_0} f\frac{DJ}{Dt}d\boldsymbol{\alpha}, \qquad (1)\end{aligned}$$

where the Jacobian $J = det(\partial X_i/\partial\alpha_j)$ is the determinant of the Jacobian matrix of the mapping $\mathbf{X}(\boldsymbol{\alpha},t)$.
How do we figure out $\frac{DJ}{Dt}$?

The first method is, choose $f = 1$, then we get

$$\frac{d}{dt}V(t) = \int_{V_0}\frac{DJ}{Dt}d\boldsymbol{\alpha}. \qquad (2)$$

Because,

$$\begin{aligned}\frac{d}{dt}V(t) &= \oint_{\partial V(t)} \underline{v}\cdot\underline{n}\,dA \\ &= \int_{V(t)} \underline{\nabla}\cdot\underline{v}\,d\mathbf{X} \\ &= \int_{V_0} \underline{\nabla}\cdot\underline{v}J\,d\boldsymbol{\alpha}. \qquad (3)\end{aligned}$$

Hence,

$$\frac{DJ}{Dt} = (\underline{\nabla}\cdot\underline{v})J. \qquad (4)$$

This yields the **Reynolds Transport Theorem**:

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} f(\mathbf{X},t)\,d\mathbf{X} &= \int_{V_0}\left[\frac{Df}{Dt} + f\underline{\nabla}\cdot\underline{v}\right]J\,d\boldsymbol{\alpha} \\ &= \int_{V(t)}(\frac{Df}{Dt} + f\underline{\nabla}\cdot\underline{v})\,d\mathbf{X} \\ &= \int_{V(t)}(\frac{\partial f}{\partial t} + \underline{\nabla}\cdot(f\underline{v}))\,d\mathbf{X} \qquad (5)\end{aligned}$$

Note that the function ‘f’ can be any kind of function, such as scalar-valued function, vector-valued function and tensor-valued function.

The second method is to calculate the derivative of the determinant $J = det(J_{ij})$ directly. If the J_{ij} are functions of time t, the derivative of the determinant is the sum of the determinants with only one column (or row) differentiated

$$\frac{DJ}{Dt} \equiv \dot{J} = \begin{vmatrix}\dot{J}_{11} & J_{12} & J_{13}\\ \dot{J}_{21} & J_{22} & J_{23}\\ \dot{J}_{31} & J_{32} & J_{33}\end{vmatrix} + \begin{vmatrix}J_{11} & \dot{J}_{12} & J_{13}\\ J_{21} & \dot{J}_{22} & J_{23}\\ J_{31} & \dot{J}_{32} & J_{33}\end{vmatrix} + \begin{vmatrix}J_{11} & J_{12} & \dot{J}_{13}\\ J_{21} & J_{22} & \dot{J}_{23}\\ J_{31} & J_{32} & \dot{J}_{33}\end{vmatrix}. \qquad (6)$$

This formula is similar to the formula for the derivative of a product: $d/dt(fgh) = (df/dt)gh + f(dg/dt)h + fg(dh/dt)$. Mathematically speaking, the derivative formula results from the fact that a determinant is a multilinear function of its rows (and columns).

All this can be written more compactly using the cofactor expansion formula for the determinant (see a book on linear algebra)

$$J \equiv det(J_{ij}) = \sum_j J_{ij}A_{ij} = \sum_i J_{ij}A_{ij} \tag{7}$$

where A_{ij} is the cofactor of the matrix element J_{ij}. The cofactor is $(-1)^{i+j}$ times the minor of that element. The minor of J_{ij} is the determinant of the n-1 by n-1 matrix obtained by deleting row i and column j from the matrix $\mathbf{J}$. The first sum (over j) gives the same determinant value for any i, the second (over i) gives the same value for any j. Another important property of determinants is that

$$\sum_j J_{kj}A_{ij} = \delta_{ki}J \tag{8}$$

where δ_{ki} is the Kronecker symbol (or tensor) =0 is $k \neq i$, = 1 if $k = i$.

Then the derivative of the determinant which is a sum of n determinants in which only one column (row) is differentiated can be written as

$$\frac{DJ}{Dt} = \sum_i \sum_j \frac{DJ_{ij}}{Dt} A_{ij}. \tag{9}$$

Note that the summations are now over all indices. This double sum is a sum of n determinants where only one of the original columns (or rows) is differentiated.

So

$$\begin{aligned} \frac{D}{Dt} Det(J_{ij}) &= \sum_i \sum_j \dot{J}_{ij} A_{ij} = \sum_i \sum_j \left(\frac{D}{Dt} \frac{\partial X_i}{\partial \alpha_j} \right) A_{ij} = \sum_i \sum_j \left(\frac{\partial v_i}{\partial \alpha_j} \right) A_{ij} \\ &= \sum_i \sum_j \left(\sum_k \frac{\partial v_i}{\partial x_k} \frac{\partial X_k}{\partial \alpha_j} \right) A_{ij} = \sum_i \sum_k \frac{\partial v_i}{\partial x_k} \sum_j \frac{\partial X_k}{\partial \alpha_j} A_{ij} \\ &= \sum_i \sum_k \frac{\partial v_i}{\partial x_k} \delta_{ki} J = \sum_k \frac{\partial v_k}{\partial x_k} J = (\underline{\nabla} \cdot \underline{v})\, J \end{aligned} \tag{10}$$