Your write-up must be clear AND concise!

1. (1) Derive the Green's function for Poisson's equation in the unbounded plane $\mathbb{R}^2$ and in 3D space $\mathbb{R}^3$.
(2) Find the Green's function for a half plane $y > 0$ with $G = 0$ at $y = 0$ and use it to solve $\boldsymbol{\nabla}^2 u = 0$, with $u(x, 0) = f(x)$.
(2) Find the Green's function for a half plane $y > 0$ with $\partial G/\partial y = 0$ at $y = 0$. [Hints: use the method of images.]

2. Consider the Helmholtz equation $\boldsymbol{\nabla}^2 u + \kappa^2 u = 0$ where κ^2 is a (positive) real number in the unbounded plane $\mathbb{R}^2$. Take a deep breath, then:
(1) Show that this equation follows from looking for eigensolutions $u(x, y, t) = \exp(\lambda t)\hat{u}(x, y)$ for both the heat equation $u_t = \nu\boldsymbol{\nabla}^2 u$ and the wave equation $u_{tt} = c^2\boldsymbol{\nabla}^2 u$. Relate κ to λ (and $\nu > 0$ or $c^2 > 0$) for both equations and (briefly) argue on physical grounds that κ^2 should indeed be a real positive number.
(2) Show that κ^2 can be removed by re-scaling the spatial dimensions as long as we are in an unbounded domain (so there is no external length scale).
(3) Find the general solution of the rescaled equation (*i.e.*with $\kappa \equiv 1$) in terms of a double Fourier integral in terms of Cartesian coordinates x and y.
(4) Transform to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and likewise for the wavevector $k_x = k\cos\alpha$, $k_y = k\sin\alpha$, where k_x and k_y are the x and y wavenumbers, respectively. Show that the complex exponentials can be combined such that $k_x x + k_y y = kr\cos(\theta - \alpha)$.
(5) Restrict your solution to axisymmetric solutions only *i.e.*$u(x, y) = v(r)$. Show that $v(r)$ satisfies Bessel's equation of order 0, therefore $v(r) = J_0(r)$. Use your Fourier approach to deduce an integral representation for $J_0(r)$.
(6) Use the method of stationary phase to deduce the leading-order asymptotic behavior of $J_0(r)$ as $r \to \infty$.