This is a Take Home Exam. You must work on it alone. You may use your class notes and the reference books.

1. Solve $u_t = u_{xx}$ in $0 < x < L$ with $u(0,t) = \partial_x u(L,t) = 0$ and $u(x,0) = f(x)$ using (a) a series solution, (b) the Green's function obtained by the method of images. Compare the two solutions (which will contain integrals involving the initial data $f(x)$).

2. Solve $u_t = u_{xx}$ in $0 < x < L$ with $u(x,0) = f(x)$ and $u(0,t) = g(t)$, $u(L,t) = 0$. Provide a general formula in terms of $f(x)$ and $g(t)$ then specify the explicit solution when $f(x) = 1$ and $g(t) = e^{-t}$.

3. Find the Green's function for the biharmonic operator $\nabla^2\nabla^2$ in the plane then solve $\nabla^2\nabla^2 u = f(x,y)$ with u and its derivatives vanishing as $x^2 + y^2 \to \infty$ and $f(x,y)$ is smooth with compact support (i.e. vanishes outside of a bounded region).

4. Solve $e^x u_x + u_y = 0$ with $u(x,0) = x$.

5. Solve the traffic flow problem $\rho_t + q_x = 0$ for initial conditions corresponding to a traffic light at $x = 0$ that turns from red to green at $t = 0$. The total number of cars that was waiting at the light is $N < \infty$. The car flux $q = \rho V$ where $V(\rho)$ is a quadratic function with $V(0) = V_{\max}$, $V'(0) = 0$ and $V(\rho_{\max}) = 0$.

6. Solve $(u_x)^2 + u_y + u = 0$ with $u(x,0) = x$.