This is a Take Home Exam. You must work on it alone. You may use your class notes and the reference books.

1. Solve $(x - x_0)\,[y'' - \alpha^2 y] = 0$ with $y \to 0$ as $|x| \to \infty$, for some constant x_0.

2. Solve $u_{xx} + u_{yy} = 0$ in the upper half plane with $u(x, 0) = f(x)$, for some sufficiently nice $f(x)$.

3. Construct a self-similar solution to the 2D heat equation $u_t = u_{xx} + u_{yy} \equiv \nabla^2 u$ such that u and its derivatives vanish as $r = (x^2 + y^2)^{1/2} \to \infty$ and u as non-zero average. Express the limit of this self-similar solution as $t \to 0^+$ in terms of generalized functions in both cartesian and polar coordinates. Find the solution to the 2D heat equation corresponding to $u(x, y, 0) = \delta(x)\delta(y)$.

4. Find a self-similar solution to Burgers' equation $u_t + uu_x = \nu u_{xx}$ where $\nu > 0$, such that u and its derivatives vanish as $|x| \to \infty$ and $\int_{-\infty}^{\infty} u(x, 0)dx = 0$. Briefly compare to the corresponding solution of the heat equation.

5. Find the first two terms in the large time asymptotic expansion of the solution to the heat equation $u_t = u_{xx}$ with $u(x, 0) = [H(x - 2\pi) - H(x - 4\pi)] \sin x$. $H(x)$ is the Heaviside step function.

6. Consider a linear PDE with dispersion relation $\omega = \omega(k)$, real, such that $d\omega/dk$ has a maximum, ω'_m say. Discuss long time asymptotics along radials $x/t = V < \omega'_m$ as well as along $x/t = \omega'_m$. Specify the amplitude decay as a function of t.