This is a Take-Home exam. You must work on this alone. You may use your class notes, the Zauderer book and the 703 book (Strang, Introduction to Applied Mathematics) but no other book. You may work on this for as long as you want but must turn it at the Thur Apr 12 lecture.

Please use Fourier transform definitions that were used in class and are summarized in the handout posted on the web page `http://kleene.math.wisc.edu/~waleffe/M704/`

1.
(a) If $\widehat{f}(k)$ is the Fourier transform of $f(x)$, show that $i\,d\widehat{f}(k)/dk$ is the Fourier transform of $xf(x)$.
(b) Use Fourier transforms to find an integral expression for the solution of the differential equation $y''(x) = xy(x)$ that satisfies $\int_{-\infty}^{\infty} y(x)dx = 1$.
(c) Solve $u_t = u_{xxx}$ with $u(x,0) = \delta(x)$.

2. The dispersion relation for gravity-capillary waves is $\omega^2 = gk + Tk^3/\rho$, where g is the acceleration of gravity, T the surface tension and ρ the water density. (a) Show that there is a minimum group velocity and find its value. Next, consider the long time asymptotics, with $x/t = O(1)$, of a disturbance initially localized near $x = 0$, e.g. $u(x,0) = \delta(x)$. (b) Show that there is a wedge in the x,t plane inside which waves are observed. (c) Modify the asymptotics to handle the minimum group velocity where $\omega'' = 0$.

3. Solve the equation $u_{tt} - u_{xx} - u_{xxt} = 0$ in $-\infty < x < \infty$, $t \geq 0$ with $u(x,0) = f(x)$, $u_t(x,0) = f''(x)$. What is the asymptotic behavior for large t?