FW Math704

This is a Take-Home exam. You must work on this alone. You may use your class notes and the Zauderer book but no other book. You may work on this for as long as you want but must turn it at the Thur Feb 22 lecture.

1. Solve the PDE $yu_x + xu_y + u = 0$ with $u(x, 0) = \exp(-x^2)$. Sketch the characteristics. What are the values of u(5, 5) and u(0, 5)?

2. Consider the traffic flow problem with $\rho(x, t)$ representing the density of cars on a highway, x is the distance along that highway and t is time. Assume that the speed of cars is a function of the density ρ only (i.e. the car and road conditions do not change in time or space) $V = V(\rho)$.

(a) Write down the PDE that controls the evolution of $\rho(x, t)$.

(b) There are two constraints on ρ and V: the density cannot be greater than ρ_{max} and the speed cannot be greater than V_{max} . Discuss very briefly why this is so. Assume $V(\rho) = a + b\rho$ (linear relationship). Find a and b in terms of ρ_{max} and V_{max} , using some plausible assumptions (e.g. $V_{\text{min}} = 0$, $\rho_{\text{min}} = 0$, cars slow down if density increases, etc...).

(c) Find a change of variables $u = f(\rho)$ that transforms the PDE into the PDE $u_t + uu_x = 0$. (d) Solve the problem $\rho(x, 0) = \rho_{\max}$ for $x \le 0$ and $\rho(x, 0) = 0$ for x > 0, which corresponds to a state where a really, really long red light at x = 0 finally turns green at t = 0. Solve this problem using BOTH the ρ equation and the u equation.

(e) Describe what kind of initial conditions lead to shocks and establish the equation that governs the shock speed.

3. Find the distance function u(x, y) to the curve $y = \cos x$ using both the method of characteristics and "Huygens' principle" which consists of finding the envelope of circles of radius u centered at $y = \cos x$. You will find the solution in parametric form. It would be nice to sketch level curves of u(x, y), you may use a plotting program to do so (e.g. Matlab). [Hint: Huygens principle is staightforward, try that first. For the method of characteristics you will need to determine u_x and u_y along $y = \cos x$.]