1. [**20pts**] Specify whether the following problems are well-posed and give the name of the equation if you recognize it. If the equation admits wave solutions, state whether or not the waves are dispersive. In all cases $-\infty < x < \infty$, $0 < t < \infty$.

Fourier analysis: $u(x,t) = e^{\sigma t} e^{ikx} \equiv e^{-i\omega t} e^{ikx}$

(a) $u_t = u_{xx}$ with $u(x,0) = f(x)$.

$\sigma = -k^2$, well-posed. Heat equation. Diffusion, no waves.

(b) $u_{tt} = u_{xx}$ with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

$\sigma \equiv -i\omega = \pm ik$, well-posed. Wave equation, not dispersive: $\omega/k = \pm 1$, $d\omega/dk = \pm 1$, all waves travel at same speed. (Two waves, one travels left, the other right, both at speed 1).

(c) $u_{ttt} = u_{xx}$ with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $u_{tt}(x,0) = h(x)$.

$\sigma^3 = -k^2$, so $\sigma = |k|^{2/3}\{-1, e^{i\pi/3}, e^{-i\pi/3}\}$, ill-posed as two of the σ's have positive real parts $= |k|^{2/3}/2$, which is unbounded as $k \to \infty$.

(d) $u_t = iu_{xx}$ with $u(x,0) = f(x)$, where $i^2 = -1$.

$\sigma = -ik^2 \equiv -i\omega$, so $\omega = k^2$. Well-posed, dispersive waves. Phase velocity $= \omega/k = k$ so short waves travel faster. This is Schrödinger's equation (quantum mechanics, particles $\equiv$ waves).

(e) $u_{tt} = u_{xx} + u$ with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

$\sigma^2 = 1 - k^2$, $\sigma = \pm\sqrt{1-k^2}$. Well-posed but unstable for $|k| < 1$. Exponential growth if $|k| < 1$ (instability), dispersive waves if $|k| > 1$.

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2. [**20pts**] Consider the PDE $u_t + u_{xx} + u_{xxxx} = 0$ with $u(x,0) = f(x)$.

(a) Give an integral representation for the solution $u(x,t)$ in $-\infty < x < \infty$, $t > 0$ and specify the asymptotic form of the solution for large t. Assume $\int_{-\infty}^{\infty} |f(x)|dx \le M < \infty$ in this case.

$u(x,t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{(k^2-k^4)t} e^{ikx} dk$, where $\hat{f}(k) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$ (Fourier transform). Now, $\int_{-\infty}^{\infty} |f(x)|dx \le M < \infty$ implies $|\hat{f}(k)| \le M/(2\pi)$. Thanks to the $e^{-k^4 t}$ the $u(x,t)$ integrand is well-behaved as $|k| \to \infty$ (for $t > 0$). (And the equation is well-posed).

For large t, the integral is increasingly dominated by the neighborhoods of $\max(k^2 - k^4)$, i.e. $k = \pm 1/\sqrt{2}$. So for large t, with $a = 1/\sqrt{2}$ and $\epsilon \ll 1$,

$$u(x,t) \approx \int_{-a-\epsilon}^{-a+\epsilon} \hat{f}(k) e^{(k^2-k^4)t} e^{ikx} dk + \int_{a-\epsilon}^{a+\epsilon} \hat{f}(k) e^{(k^2-k^4)t} e^{ikx} dk$$

$$\approx \left(\hat{f}(-a) e^{-iax} + \hat{f}(a) e^{iax}\right) e^{t/4} \int_{-\infty}^{\infty} e^{-2s^2 t} e^{isx} ds$$

$$= \left(\hat{f}(-a) e^{-iax} + \hat{f}(a) e^{iax}\right) \sqrt{\frac{\pi}{2t}}\, e^{t/4} e^{-x^2/8t}.$$

(b) What is the solution and its asymptotic form for large t if $f(x) = f(x + 2\pi)$?
Function is periodic, so its Fourier transform does not exist (it's a sum of "delta functions"), but its Fourier series does $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$, n integer, and $C_n = (2\pi)^{-1} \int_0^{2\pi} f(x) e^{-inx} dx$. $u(x,t) = \sum_{n=-\infty}^{\infty} C_n e^{inx} e^{(n^2-n^4)t}$. For large t, $u(x,t) \approx C_0 + C_1 e^{ix} + C_{-1} e^{-ix}$, steady! no exponential growth as in the unbounded case.

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3. [10pts] Solve $u_{tt} = u_{xx} + \delta(x - Vt)$ with $u(x,t)$, $u_t(x,t) \to 0$ as $t \to -\infty$ and $V > 1$. Comment on the case $V = 1$.
The Green's function of the 1D wave equation is (cf. class notes)

$$G(x,t;x_0,t_0) = \frac{1}{2} H(t - t_0) \left[H(x - x_0 + t - t_0) - H(x - x_0 - t + t_0) \right].$$

So the solution to $u_{tt} = u_{xx} + F(x,t)$ with $u(x,t)$, $u_t(x,t) \to 0$ as $t \to -\infty$ is

$$u(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x_0,t_0) G(x,t;x_0,t_0) dx_0 dt_0.$$

Here $F(x_0,t_0) = \delta(x_0 - Vt_0)$, so

$$u(x,t) = \frac{1}{2} \int_{-\infty}^{t} \left[H(x - Vt_0 + t - t_0) - H(x - Vt_0 - t + t_0) \right] dt_0.$$

Note that $x - Vt_0 + t - t_0 \geq x - Vt_0 - t + t_0$, $\forall t \geq t_0$. Now $H(a) = 0$ if $a < 0$, $H(a) = 1$ if $a > 0$. So the integral vanishes unless

$$x - Vt_0 + t - t_0 > 0 > x - Vt_0 - t + t_0,$$

or, using $V > 1$,

$$\frac{x+t}{V+1} > t_0 > \frac{x-t}{V-1}.$$

This requires

$$\frac{x+t}{V+1} > \frac{x-t}{V-1} \iff Vt > x.$$

In that case, $t > (x + t)/(V + 1)$ and

$$u(x,t) = \frac{1}{2} \int_{(x-t)/(V-1)}^{(x+t)/(V+1)} dt_0 = \frac{1}{2} \left(\frac{x+t}{V+1} - \frac{x-t}{V-1} \right) = \frac{Vt - x}{V^2 - 1},$$

if $Vt > x$ and $u(x,t) = 0$ if $Vt < x$.
This problem could be solved as a homogeneous IVP with appropriate initial conditions given on the line $x = Vt$. If $V = 1$, this is equivalent to imposing initial conditions along a characteristic curve, and that's not good.