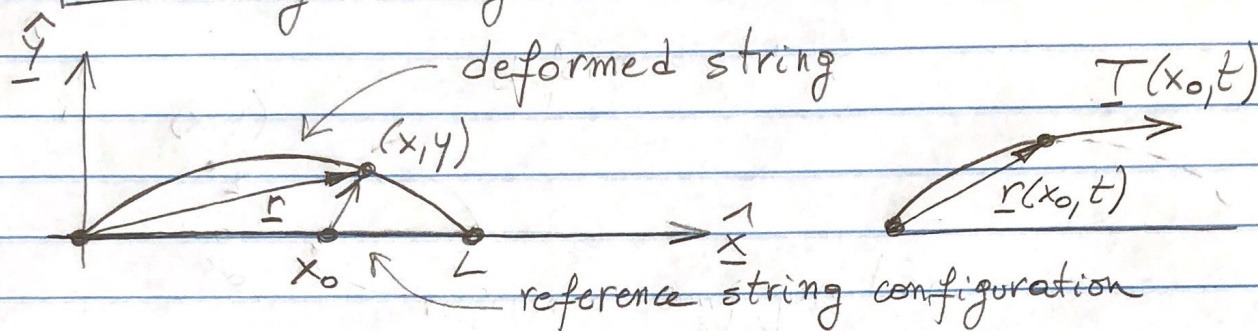


Vibrating string

better derivation!



- string point $(x_0, 0) \rightarrow (x, y)$ at time t
All x_0 in $0 \leq x_0 \leq L$

$$\underline{r}(x_0, t) = x \hat{x} + y \hat{y} \quad \Rightarrow \begin{cases} x = x(x_0, t) = x_0 + u(x_0, t) \\ y = y(x_0, t) = v(x_0, t) \end{cases}$$

- Tension $\underline{T}(x_0, t)$ = force at (x, y) on $0 \rightarrow x_0$ material

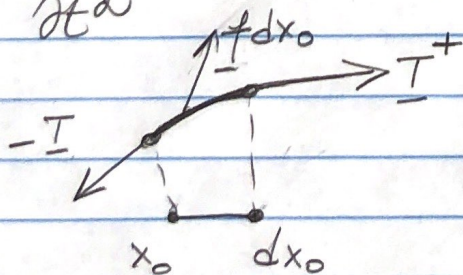
$$\underline{T}(x_0, t) = T(x_0, t) \underline{\hat{t}}(x_0, t) \quad \underline{\hat{t}} = \text{unit tangent to string}$$

$$\underline{\hat{t}} = \frac{\partial \underline{r} / \partial x_0}{|\partial \underline{r} / \partial x_0|} = \frac{(1 + \partial u / \partial x_0) \hat{x} + (\partial v / \partial x_0) \hat{y}}{\sqrt{(1 + \partial u / \partial x_0)^2 + (\partial v / \partial x_0)^2}}$$

Newton:

- $m \underline{a} = \underline{F}$ for a material section $[x_0, x_0 + dx_0]$

$$\rightarrow \rho dx_0 \frac{\partial^2 \underline{r}}{\partial t^2} = \underline{T}(x_0 + dx_0, t) - \underline{T}(x_0, t) + \underline{f} dx_0$$



ρ = mass per unit length
 $\underline{f}$ = force per unit length

Vibrating string

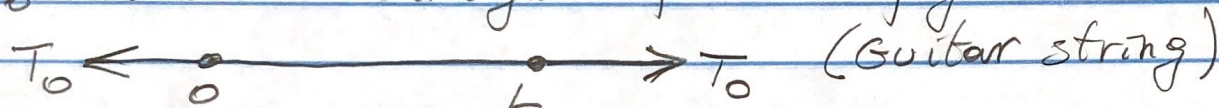
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(2)

$$\Rightarrow \left[\rho \frac{\partial^2 \underline{r}}{\partial t^2} = \frac{\partial}{\partial x_0} \left(\frac{T \left((1 + \frac{\partial u}{\partial x_0}) \underline{\hat{x}} + \frac{\partial v}{\partial x_0} \underline{\hat{y}} \right)}{\sqrt{(1 + \frac{\partial u}{\partial x_0})^2 + (\frac{\partial v}{\partial x_0})^2}} \right) + \underline{f} \right]$$

Now $\underline{r} = (x_0 + u) \underline{\hat{x}} + v \underline{\hat{y}}$, $T = T_0 + \Delta T$, T_0 constant

T_0 = tension in straight reference configuration



Linearize: assume $|\frac{\partial u}{\partial x_0}| \ll 1$, $|\frac{\partial v}{\partial x_0}| \ll 1$, $|\Delta T| \ll T_0$

$$\Rightarrow (1 + \frac{\partial u}{\partial x_0})^2 \rightarrow 1, \quad 1 + (\frac{\partial v}{\partial x_0})^2 \rightarrow 1, \quad (T_0 + \Delta T) \frac{\partial v}{\partial x_0} \rightarrow T_0 \frac{\partial v}{\partial x_0}$$

separating x & y with $\underline{r} = (x_0 + u) \underline{\hat{x}} + v \underline{\hat{y}}$

$$\Rightarrow \left[\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= \frac{\partial (\Delta T)}{\partial x_0} + \underline{\hat{x}} \cdot \underline{f} \\ \rho \frac{\partial^2 v}{\partial t^2} &= T_0 \frac{\partial^2 v}{\partial x_0^2} + \underline{\hat{y}} \cdot \underline{f} \end{aligned} \right]$$

u, v decoupled

- u eqn. requires knowing constitutive law, that is how ΔT depends on u . Elastic: $\Delta T = k \frac{\partial u}{\partial x_0}$

$$\Rightarrow \left[\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x_0} \left(k \frac{\partial u}{\partial x_0} \right) + \underline{\hat{x}} \cdot \underline{f} \right]$$

- v eqn. does not need constitutive law because $T_0 \gg |\Delta T|$

$$\Rightarrow \left[\rho \frac{\partial^2 v}{\partial t^2} = T_0 \frac{\partial^2 v}{\partial x_0^2} + \underline{\hat{y}} \cdot \underline{f} \right]$$