

A Sturm-Liouville differential equation reads

$$\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + (\lambda w(x) - q(x)) y = 0. \quad (1)$$

where λ is a constant (Haberman Chap. 5 & 7).

1. If $p(x)$, $q(x)$ and $w(x) \geq 0$ in $a \leq x \leq b$ with $p(a) = 0 = p(b)$ and $y(x)$ bounded as $x \rightarrow a$ and $x \rightarrow b$, prove that λ is real and positive. If $p(a) = 0$ but $p(b) > 0$, what linear boundary conditions at $x = b$ guarantee that $\lambda > 0$? Prove. State and prove the eigenfunction orthogonality.

2. Transform

$$a(x) \frac{d^2 y}{dx^2} + b(x) \frac{dy}{dx} + c(x) y = 0$$

into Sturm-Liouville form. What are the restrictions on $a(x)$? (Haberman exercise 5.3.3)

3. The *Legendre equation* is

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \lambda y = 0. \quad (2)$$

Put it into Sturm-Liouville form. What are $p(x)$, $q(x)$, $w(x)$? Show that $y(x)$ bounded at $x = \pm 1$ requires $\lambda = n(n+1)$ for any integer $n > 0$, and that the corresponding eigenfunctions $y_n(x)$ are (Legendre) polynomials of degree n . What is the orthogonality relation for $y_n(x)$?

4. The *Chebyshev equation* is

$$(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + \lambda y = 0, \quad -1 \leq x \leq 1. \quad (3)$$

Determine the asymptotic behavior of $y(x)$ as $x \rightarrow \pm 1$. Show that $y(x)$ bounded at $x = \pm 1$ requires $\lambda = n^2$ for any integer $n > 0$, and that the eigenfunctions are (Chebyshev) polynomials. Write the equation in Sturm-Liouville form. What are $p(x)$, $q(x)$, $w(x)$? What is the eigenfunction orthogonality relation?

5. Take k derivatives of (2) to show that the functions $\Phi^{(k)}(x) = d^k y / dx^k$ satisfy

$$(1 - x^2) \frac{d^2 \Phi^{(k)}}{dx^2} - 2(k+1)x \frac{d\Phi^{(k)}}{dx} + (\lambda - k(k+1)) \Phi^{(k)} = 0. \quad (4a)$$

Show that the Sturm-Liouville form is

$$\frac{d}{dx} \left((1 - x^2)^{k+1} \frac{d\Phi^{(k)}}{dx} \right) + (\lambda - k(k+1)) (1 - x^2)^k \Phi^{(k)} = 0. \quad (4b)$$

Deduce that the $\Phi^{(k)}(x)$ are orthogonal polynomials with weight $w(x) = (1 - x^2)^k$, hence *the derivatives of Legendre polynomials are orthogonal polynomials* (and similarly for the derivatives of *Chebyshev polynomials* (3)). Discuss why the eigenvalues are still $\lambda = n(n+1)$ but only for $n \geq k$. What is orthogonality relation for the $\Phi_n^{(k)}(x)$? The *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (5)$$

follows from ‘telescopic combination’ of derivatives of the equations (4b) for $k = n - 1 \rightarrow 0$.

6. Consider the associated Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(\mu - \frac{m^2}{1-x^2}\right)y = 0. \quad (6)$$

Determine the asymptotic behavior of $y(x)$ as $x \rightarrow \pm 1$. Let $y(x) = (1-x^2)^{m/2}W(x)$ and show that

$$(1-x^2)\frac{d^2W}{dx^2} - 2(m+1)x\frac{dW}{dx} + (\mu - m(m+1))W = 0, \quad (7)$$

and therefore $\mu = n(n+1)$ for $W(\pm 1)$ bounded, with $W(x) = \Phi_n^{(m)}(x) = d^m P_n(x)/dx^m$, and the bounded eigenfunctions of (6) are

$$y(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m} \equiv P_n^m(x). \quad (8)$$

7. Consider

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + (x^2 - \mu)y = 0, \quad (9)$$

for $x \geq 0$. Determine the asymptotic behavior of $y(x)$ as $x \rightarrow 0^+$.

Let $y(x) = x^\alpha Z(x)$ and find α such that $Z(x)$ satisfies *Bessel's equation*

$$x^2 \frac{d^2Z}{dx^2} + x \frac{dZ}{dx} + (x^2 - \nu^2)Z = 0, \quad (10)$$

with $\nu^2 = \mu + 1/4$. Conclude that $Z(x) = J_{n+1/2}(x)$, $Y_{n+1/2}(x)$ when $\mu = n(n+1)$.

8. Solve

$$\frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

for $u(r, t)$ in $0 \leq a \leq r \leq b$, $t > 0$ with

$$\frac{\partial u}{\partial r}(a, t) = 0, \quad u(b, t) = 0 \quad \forall t$$

and $u(r, 0) = F(r)$. What is the solution if $a = 0$?

9. Consider Laplace's equation $\nabla^2 u = 0$ in cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$. Look for solutions of the form

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n f_n(\theta),$$

for some $a > 0$. Find $f_n(\theta)$.

10. Consider Laplace's equation $\nabla^2 u = 0$ in spherical coordinates $x = r \cos \theta$, $y = r \sin \theta \cos \varphi$, $z = r \sin \theta \sin \varphi$. Look for solutions of the form

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n f_n(\theta, \varphi),$$

for some $a > 0$. Find $f_n(\theta, \varphi)$.

11. Discuss canvas-posted solutions to Haberman problems 7.9.1d, 7.10.2b, 7.10.9a. Hide solutions, derive all by yourself.