

1 Orthogonality of cosine, sine and complex exponentials

The functions $\cos n\theta$ form a complete orthogonal basis for piecewise C^1 functions in $0 \leq \theta \leq \pi$,

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = \frac{\pi}{2} \delta_{mn} (1 + \delta_{n0}) = \begin{cases} \pi & \text{if } m = n = 0 \\ \pi/2 & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases} \quad (1)$$

The functions $\sin n\theta$ form a complete orthogonal basis for piecewise C^1 functions in $0 \leq \theta \leq \pi$,

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = \frac{\pi}{2} \delta_{mn} (1 - \delta_{n0}) = \begin{cases} 0 & \text{if } m = n = 0 \\ \pi/2 & \text{if } m = n \neq 0 \\ 0 & \text{if } m \neq n \end{cases} \quad (2)$$

The functions $\cos n\theta$ can only represent *even* functions in $-\pi \leq \theta \leq \pi$. The functions $\sin n\theta$ can only represent *odd* functions in $-\pi \leq \theta \leq \pi$. So *both* $\cos n\theta$ and $\sin n\theta$ are needed to represent arbitrary piecewise C^1 functions in $-\pi \leq \theta \leq \pi$, in that case it is more convenient to work with complex exponentials $\exp(in\theta)$ that are complex orthogonal in $-\pi \leq \theta \leq \pi$

$$\int_{-\pi}^\pi e^{-im\theta} e^{in\theta} \, d\theta = 2\pi \delta_{mn}. \quad (3)$$

2 Fourier series

Thus a piecewise smooth function $f(\theta)$ can be represented as a Fourier cosine series

$$f(\theta) \approx \sum_{n=0}^{\infty} A_n \cos n\theta \equiv C(\theta) \quad (4a)$$

with

$$A_0 = \frac{1}{\pi} \int_0^\pi f(\theta) \, d\theta, \quad A_n = \frac{2}{\pi} \int_0^\pi f(\theta) \cos n\theta \, d\theta. \quad (4b)$$

The same function $f(\theta)$ can also be represented as a Fourier sine series

$$f(\theta) \approx \sum_{n=1}^{\infty} B_n \sin n\theta \equiv S(\theta) \quad (5a)$$

with

$$B_n = \frac{2}{\pi} \int_0^\pi f(\theta) \sin n\theta \, d\theta. \quad (5b)$$

Although these expansions will match in $0 < \theta < \pi$, they will differ outside that interval since the cosine series (4a) is necessarily even $C(\theta) = C(-\theta)$ while the sine series is odd $S(\theta) = -S(-\theta)$. The function $f(\theta)$ may not be even or odd, for example $f(\theta) = \theta + \theta^2$ is neither odd or even. The cosine and sine series are also necessarily periodic of period 2π , $C(\theta + 2\pi) = C(\theta)$, $S(\theta + 2\pi) = S(\theta)$. The function $f(\theta)$ is not necessarily periodic (see also Haberman figs. 3.3.15 and 3.3.16).

The full Fourier series of $f(\theta)$ is

$$f(\theta) \approx \sum_{n=-\infty}^{\infty} C_n e^{in\theta} \equiv F(\theta) \quad (6a)$$

with

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta. \quad (6b)$$

Substituting $e^{in\theta} = \cos n\theta + i \sin n\theta$ in the full Fourier series (6a) and combining the $\cos n\theta$ and $\sin n\theta$ terms yields

$$f(\theta) \approx \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = F(\theta) \quad (7a)$$

with

$$a_0 = C_0, \quad a_n = C_n + C_{-n}, \quad b_n = i(C_n - C_{-n}), \quad (7b)$$

and from (6b) with $e^{-in\theta} + e^{in\theta} = 2 \cos n\theta$ and $i(e^{-in\theta} - e^{in\theta}) = 2 \sin n\theta$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta = \frac{1}{\pi} \int_0^{\pi} \frac{f(\theta) + f(-\theta)}{2} d\theta \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta = \frac{2}{\pi} \int_0^{\pi} \frac{f(\theta) + f(-\theta)}{2} \cos n\theta d\theta \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta = \frac{2}{\pi} \int_0^{\pi} \frac{f(\theta) - f(-\theta)}{2} \sin n\theta d\theta. \end{aligned} \quad (7c)$$

Thus $a_n = A_n$, $b_n = 0$ if $f(\theta) = f(-\theta)$ (even), and $a_n = 0$, $b_n = B_n$ if $f(\theta) = -f(-\theta)$ (odd).

3 Examples

As calculated in class from (4b), (5b), for the even function $f(\theta) = 1$ (see also Haberman 3.3.10)

$$1 \approx \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)\theta}{2m-1}, \quad (8a)$$

$$1 = 1 + \sum_{n=1}^{\infty} 0 \cos n\theta, \quad (8b)$$

and for the odd function $f(\theta) = \theta$ (see also Haberman 3.3.11,12,21,22,23)

$$\theta \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta, \quad (9a)$$

$$\theta \approx \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos(2m-1)\theta}{(2m-1)^2} \quad (9b)$$

Now $d\theta/d\theta = 1$, and the derivative of the cosine series (9b) for θ indeed yields the sine series (8a) for 1, but the term-by-term derivative of the slowly decaying (9a)

$$\frac{d}{d\theta} \left(2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta \right) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos n\theta \neq 1 \quad (10)$$

is a *non-decaying* cosine series that does not equal the cosine series (8b) for $f(\theta) = 1$ (see also Haberman section 3.4, counterexample). The cosine series (10) converges extremely slowly towards 1 for $\theta \neq (2k-1)\pi$ and goes to $-\infty$ at $\theta = (2k-1)\pi$, for any integer k .

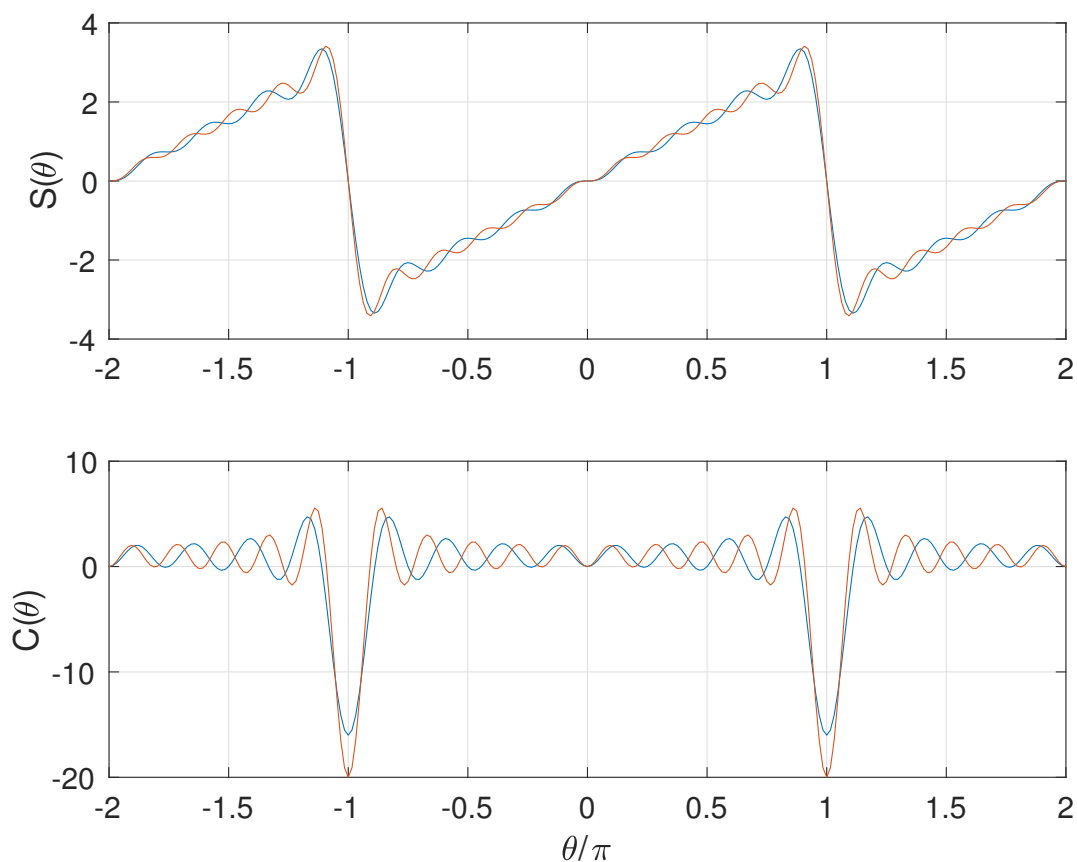


Figure 1: Sine and cosine series (9a), (10) truncated to 8 and 10 terms

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%% Matlab Script for plots
%% Waleffe, Math 322, UW-Madison 2019/10/29
t=2*pi*[-1:1/2^7:1];
C=zeros(size(t)); S=zeros(size(t));
figure(1), clf
for n=1:10,
    C=C+2*(-1)^(n+1)*cos(n*t);
    S=S+2*(-1)^(n+1)*sin(n*t)/n;
    if (n==8)|(n==10)
        subplot(2,1,1),plot(t/pi,S), hold on
        subplot(2,1,2),plot(t/pi,C), hold on
    end
end
subplot(2,1,1), h=gca; set(h,'fontsize',14), grid
subplot(2,1,1), ylabel('S(\theta)', 'fontsize',16)
subplot(2,1,2), h=gca; set(h,'fontsize',14), grid
subplot(2,1,2), xlabel('\theta/\pi', 'fontsize',16)
subplot(2,1,2), ylabel('C(\theta)', 'fontsize',16)
print -depsc SinCos.eps

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Differentiation term-by-term reduces the rate of convergence of the series, and may prevent con-

vergence since differentiation of series multiplies the coefficients by n : $A_n \rightarrow -nA_n = B'_n$, $B_n \rightarrow nB_n = A'_n$, where A'_n , B'_n are the cosine and sine series coefficients of the derivative, respectively.

Integration term-by-term accelerates convergence of the series since integration divides each coefficient by a factor of n , $(A_n, B_n) \rightarrow (A_n, -B_n)/n$, but integration introduces constants of integration. For example, the integrals of (9a) and (9b) that vanish at $\theta = 0$ are

$$\frac{\theta^2}{2} = \int_0^\theta \theta \, d\theta \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (\cos n\theta - 1), \quad (11a)$$

$$\frac{\theta^2}{2} = \int_0^\theta \theta \, d\theta \approx \frac{\pi}{2}\theta - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)\theta}{(2m-1)^3}. \quad (11b)$$

The first one is a cosine series for $\frac{1}{2}\theta^2 = \sum_n A_n \cos n\theta$, with constant term $A_0 = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$, but that term is simply the average of $\theta^2/2$ over $(0, \pi)$, that is

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = A_0 = \frac{1}{\pi} \int_0^\pi \frac{\theta^2}{2} \, d\theta = \frac{\pi^2}{6},$$

thus (11a) yields the cosine series

$$\frac{\theta^2}{2} \approx \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta. \quad (12)$$

The sine series for $\theta^2/2$ follows from (11b) with (9a) for θ , yielding

$$\begin{aligned} \frac{\theta^2}{2} &\approx \pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)\theta}{(2m-1)^3} \\ &= \sum_{m=1}^{\infty} \left(\frac{\pi}{2m-1} - \frac{4}{\pi(2m-1)^3} \right) \sin(2m-1)\theta - \frac{\pi}{2m} \sin 2m\theta. \end{aligned} \quad (13)$$

Of course, the cosine coefficients A_n in (12) and B_n in (13) can also be computed directly using two integration by parts from

$$A_n = \frac{2}{\pi} \int_0^\pi \frac{\theta^2}{2} \cos n\theta \, d\theta, \quad B_n = \frac{2}{\pi} \int_0^\pi \frac{\theta^2}{2} \sin n\theta \, d\theta.$$

Finally, the sine series for $\theta(1 - \theta/\pi)$ follows directly from (11b), since

$$\frac{\pi}{2}\theta \left(1 - \frac{\theta}{\pi}\right) = \frac{\pi}{2}\theta - \frac{\theta^2}{2} \approx \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)\theta}{(2m-1)^3}. \quad (14)$$

For $\theta = \pi x/L$, this yields

$$\frac{\pi^2}{2L}x \left(1 - \frac{x}{L}\right) \approx \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin((2m-1)\pi x/L)}{(2m-1)^3}. \quad (15)$$

4 Convolution

Could we also obtain the Fourier series of θ^2 by squaring that for θ ? Yes, but this involves *convolution* products, and that is easier to analyze in the case of full Fourier series.

Given two Fourier series $F(\theta)$ and $G(\theta)$,

$$F(\theta) = \sum_{n=-\infty}^{\infty} \hat{F}_n e^{in\theta}, \quad G(\theta) = \sum_{n=-\infty}^{\infty} \hat{G}_n e^{in\theta}, \quad (16)$$

what is the Fourier series of their product $H(\theta) = F(\theta)G(\theta)$? Multiplying the two series, we find

$$H(\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{F}_m \hat{G}_n e^{i(m+n)\theta} = \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \hat{F}_{k-n} \hat{G}_n \right) e^{ik\theta} \equiv \sum_{k=-\infty}^{\infty} \hat{H}_k e^{ik\theta}$$

thus the Fourier coefficients of $H(\theta) = F(\theta)G(\theta)$ are given by the *convolution sum*

$$\boxed{\hat{H}_k = \sum_{n=-\infty}^{\infty} \hat{F}_{k-n} \hat{G}_n = \sum_{m=-\infty}^{\infty} \hat{F}_m \hat{G}_{k-m}.} \quad (17)$$

For example, the Fourier sine series (9a) for θ is

$$\theta \approx 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta = i \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (e^{in\theta} - e^{-in\theta}) = i \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n}{n} e^{in\theta}$$

so $\theta^2 \approx \sum_{k=-\infty}^{\infty} \hat{H}_k e^{ik\theta}$ with from (17)

$$\hat{H}_k = (-1)^{k+1} \sum_{\substack{n=-\infty \\ n \neq 0, k}}^{\infty} \frac{1}{n(k-n)}.$$

That $\hat{H}_k$ is a bit scary, but it is obviously real, and it must equal $\hat{H}_{-k}$ since the Fourier series adds up to the real θ^2 . Now

$$\frac{1}{n(k-n)} = \frac{1}{k} \left(\frac{1}{n} + \frac{1}{k-n} \right),$$

then, boldly,

$$\sum_{\substack{n=-\infty \\ n \neq 0, k}}^{\infty} \frac{1}{n(k-n)} = \frac{1}{k} \sum_{\substack{n=-\infty \\ n \neq 0, k}}^{\infty} \frac{1}{n} + \frac{1}{k} \sum_{\substack{n=-\infty \\ n \neq 0, k}}^{\infty} \frac{1}{k-n} = -\frac{2}{k^2}$$

because $\sum_{n \neq 0} n^{-1} = 1 - 1 + 1/2 - 1/2 + \dots = 0$ so $\sum_{n \neq 0, k} n^{-1} = 0 - 1/k$, and likewise for the other sum (maybe after substitution $m = k - n$). Thus

$$\hat{H}_k = 2 \frac{(-1)^k}{k^2} = \hat{H}_{-k}$$

and

$$\theta^2 \approx \sum_{k=-\infty}^{\infty} \hat{H}_k e^{ik\theta} = \hat{H}_0 + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos k\theta,$$

with $\hat{H}_0 = \pi^{-1} \int_0^\pi \theta^2 d\theta = \pi^2/3 = 2 \sum_{n=1}^{\infty} n^{-2}$, thus recovering (12), though not the easiest way! But the general convolution result (17) is important for Fourier series. (It must be, it's boxed!)