

Series solutions of ODEs

As discussed in class, a non-singular differential equation such as

$$\frac{d^2 w}{dz^2} = w, \quad (1)$$

can be solved by representing the unknown function $w(z)$ as a Taylor series

$$w(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (2)$$

Substituting the series (2) into the differential equation (1) yields recurrence relations for the coefficients a_n but leaves a_0 and a_1 as undetermined ‘constants of integration’. This conveniently yields two linearly independent solutions, $w_0(z)$ even and $w_1(z)$ odd, defined from the choices

$$\begin{aligned} (a_0, a_1) = (1, 0) &\Rightarrow w_0(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!} \equiv \cosh z, \\ (a_0, a_1) = (0, 1) &\Rightarrow w_1(z) = \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m+1)!} \equiv \sinh z. \end{aligned}$$

Bessel’s equation of order ν ,

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0 \quad (3)$$

has a *regular singular point* at $z = 0$. It behaves like the equidimensional ODE

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - \nu^2 w = 0$$

near $z = 0$, and has solutions of the form $w(z) \sim z^r$, with $r = \pm\nu$, as $z \rightarrow 0$. For $\nu = 0$, the asymptotic behavior is $w(z) \sim 1$ and $w(z) \sim \ln z$. Bessel’s equation therefore does not have simple Taylor series solutions $w(z) = \sum_{n=0}^{\infty} a_n z^n$ about $z = 0$, in general, but it has solutions of the form

$$w(z) = z^r \sum_{n=0}^{\infty} a_n z^n. \quad (4)$$

Exercises

1. Derive the recurrence relations for the coefficients a_n for Bessel’s equation (3). As *Ruifu* was saying in class, show that the choices $(a_0, a_1) = (1, 0)$ and $(0, 1)$ for the same value of r , yield the *same* solution in this Bessel case.
2. Show that if ν is *not* an integer, then the two values $r = \pm\nu$ yield two linearly independent solutions of Bessel’s equation. Find those series for both r ’s with $a_0 \neq 0$, $a_1 = 0$. Show your work. What happens to the coefficients of your series as $\nu \rightarrow m$ integer?
3. Show that if $\nu = m \geq 0$ is an integer, then the recurrence relations for $r = -m$ is singular for $n = 2m$, thereby forcing the series for $r = -m$ to reduce to that for $r = m \geq 0$, and failing to provide a 2nd linearly independent solution.

The degeneracy when ν is an integer is similar to that occurring for constant coefficient ODEs when the roots r_1, r_2 of the characteristic equation for exponential solutions e^{rx} are repeated, $r_1 = r_2$. For example for

$$w'' + 2w' + w = 0,$$

whose characteristic equation is $(r+1)^2 = 0$. That case can be handled by perturbing the problem so the roots r_1, r_2 are distinct, for example by considering

$$w'' + 2w' + \nu^2 w = 0$$

for $\nu \neq 1$, with roots $r_1, r_2 = -1 \pm \sqrt{1 - \nu^2}$, and redefining the 2nd solution as a linear combination of the two linearly independent solutions $e^{r_1 x}$ and $e^{r_2 x}$ that does not vanish or reduce to $e^{r_1 x}$ in the limit $\nu \rightarrow 1$. That definition with a *distinguished limit* is

$$\frac{e^{r_2 x} - e^{r_1 x}}{r_2 - r_1} \rightarrow x e^{r_1 x}.$$

For Bessel's equation (3), the series expansion yields two linearly independent solutions when ν is not an integer. They are defined as $J_\nu(z)$ and $J_{-\nu}(z)$, such that $a_1 = 0$ and

$$a_0 = \frac{1}{2^r \Gamma(r+1)}, \quad (5)$$

for $r = \pm\nu$, where

$$\Gamma(r+1) = \int_0^\infty t^r e^{-t} dt$$

is the *Gamma function*, a generalization of the factorial to non-integer arguments.

A second solution of Bessel's equation with a distinguished limit as $\nu \rightarrow m$ integer is

$$Y_\nu(z) = \frac{\cos \nu\pi}{\sin \nu\pi} \frac{J_\nu(z) - J_{-\nu}(z)}{2}. \quad (6)$$

Exercises

4. Show that $\Gamma(r+1) = r\Gamma(r)$ and $\Gamma(1) = 1$, thus $\Gamma(m+1) = m!$ for integer $m \geq 0$.
5. Show that $\Gamma(r+1) \rightarrow \infty$ as $r \rightarrow -1^+$. In fact, $\Gamma(r+1) \rightarrow \pm\infty$ as $r \rightarrow -m$ for any positive integer m , but you do not need to show that.
6. Show that with the normalization (5)

$$\lim_{\nu \rightarrow m} J_{-\nu}(z) = J_{-m}(z) = (-1)^m J_m(z).$$