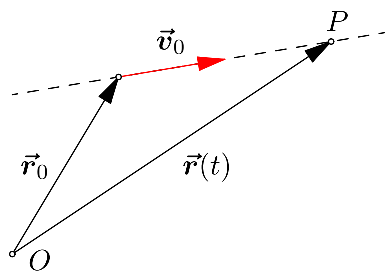
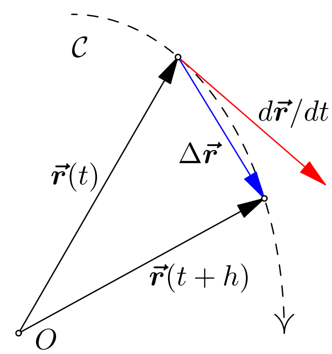
1 Curves, Surfaces, Volumes and their integrals

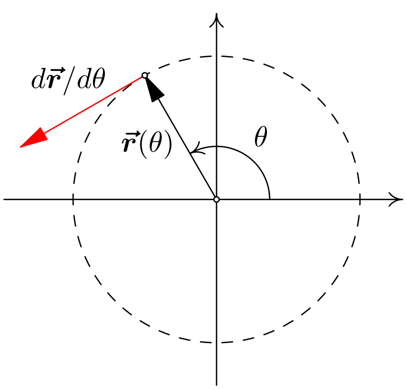
1.1 Curves



Recall the parametric equation of a line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}_0$, where $\vec{r}(t) = \overrightarrow{OP}$ is the position vector of a point P on the line with respect to some 'origin' O, $\vec{r}_0$ is the position vector of a reference point on the line and $\vec{v}_0$ is a vector parallel to the line. Note that this can be interpreted as the linear motion of a particle with constant velocity $\vec{v}_0$ that was at the point $\vec{r}_0$ at time $t = 0$ and $\vec{r}(t)$ is the position at time t.



More generally, a vector function $\vec{r}(t)$ of a real variable t defines a curve $\mathcal{C}$. The vector function $\vec{r}(t)$ is the *parametric representation* of that curve and t is the parameter. It is useful to think of t as time and $\vec{r}(t)$ as the position of a particle at time t. The collection of all the positions for a range of t is the particle trajectory. The vector $\Delta\vec{r} = \vec{r}(t+h) - \vec{r}(t)$ is a *secant vector* connecting two points on the curve, if we divide $\Delta\vec{r}$ by h and take the limit as $h \to 0$ we obtain the vector $d\vec{r}/dt$ which is *tangent* to the curve at $\vec{r}(t)$. If t is time, then $d\vec{r}/dt = \vec{v}$ is the velocity.



The parameter can have any name and does not need to correspond to time. For instance $\vec{r}(t) = \hat{\boldsymbol{x}}\, a\cos\theta(t) + \hat{\boldsymbol{y}}\, a\sin\theta(t)$, with $\theta(t) = \pi\cos t$ and $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ orthonormal, is the position of a particle moving around a circle of radius a, but the particle oscillates back and forth around the circle. That same circle is more simply described by

$$\vec{r}(\theta) = \hat{\boldsymbol{x}}\, a\cos\theta + \hat{\boldsymbol{y}}\, a\sin\theta,$$

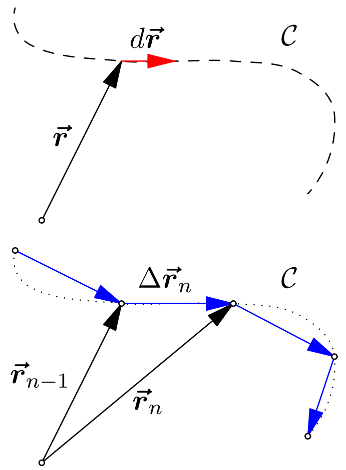
where θ is a real parameter that can be interpreted as the angle between the position vector and the $\hat{\boldsymbol{x}}$ basis vector.

Remark: Note the common abuse of notation where $\vec{r}(t)$ and $\vec{r}(\theta)$ are two *different* vector functions. Both represent points on the same curve but not the same points, for instance $\vec{r}(t = \pi/2) = a\hat{\boldsymbol{x}}$ while $\vec{r}(\theta = \pi/2) = a\hat{\boldsymbol{y}}$. In principle, we should distinguish between the two vector functions writing $\vec{r} = \vec{f}(t)$ and $\vec{r} = \vec{g}(\theta)$ with $\vec{f}(t) = \vec{g}(\theta(t))$, but in applications this quickly becomes cumbersome and a nice advantage of this abuse of notation is that it fits nicely with the *chain rule*

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{d\theta}\frac{d\theta}{dt}.$$

Remark: A convenient parametrization, conceptually, is to use distance along the curve as the parameter, often denoted s and called the *arclength*. Once a curve is specified and an $s = 0$ reference point has been chosen then the distance s from that reference point along the curve uniquely defines a point on the curve, *e.g.* for the highway 90 curve, s could be picked as distance from Chicago, positive westward. For the circle of radius a, this could be the parametrization $\vec{r}(s) = \hat{\boldsymbol{x}} a\cos(s/a) + \hat{\boldsymbol{y}} a\sin(s/a)$, where s is now the distance along the circle from the x-axis and s/a is the angle in radians.

1.2 Integrals along curves, or 'line integrals'



Line element: Given a curve $\mathcal{C}$, the *line element* denoted $d\vec{r}$ is an 'infinitesimal' secant vector. This is a useful shortcut for the procedure of approximating the curve by a succession of secant vectors $\Delta\vec{r}_n = \vec{r}_n - \vec{r}_{n-1}$ where $\vec{r}_{n-1}$ and $\vec{r}_n$ are two consecutive points on the curve, with $n = 1, 2, \ldots, N$ integer, then taking the limit $\max|\Delta\vec{r}_n| \to 0$ (so $N \to \infty$). In that limit, the direction of the secant vector $\Delta\vec{r}_n$ becomes identical with that of the tangent vector at that point. If an explicit parametric representation $\vec{r}(t)$ is known for the curve then

$$d\vec{r} = \frac{d\vec{r}(t)}{dt}\,dt \tag{1}$$

The typical *'line' integral* along a curve $\mathcal{C}$ has the form $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$ where $\vec{F}(\vec{r})$ is a vector field, *i.e.* a vector function of position. If $\vec{F}(\vec{r})$ is a force, this integral represent the net work done by the force on a particle as the latter moves along the curve. We can make sense of this integral as the *limit of a sum*, namely breaking up the curve into a chain of N secant vectors $\Delta\vec{r}_n$ as above then

$$\boxed{\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \lim_{|\Delta\vec{r}_n|\to 0} \sum_{n=1}^{N} \vec{F}(\vec{r}_n)\cdot\Delta\vec{r}_n} \tag{2}$$

This also provides a practical procedure to estimate the integral by approximating it by a finite sum. A better approximation would be to use the *trapezoidal rule*, replacing $\vec{F}(\vec{r}_n)$ in the sum by $\frac{1}{2}\left(\vec{F}(\vec{r}_n) + \vec{F}(\vec{r}_{n-1})\right)$, or the *midpoint* rule replacing $\vec{F}(\vec{r}_n)$ by $\vec{F}\left(\frac{1}{2}(\vec{r}_n + \vec{r}_{n-1})\right)$. These finite sums converge to the same limit, $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r}$, but faster for nice functions.
If an explicit representation $\vec{r}(t)$ is known then we can reduce the line integral to a regular Math 221 integral:

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_{t_a}^{t_b} \left(\vec{F}(\vec{r}(t))\cdot\frac{d\vec{r}(t)}{dt}\right) dt, \tag{3}$$

where $\vec{r}(t_a)$ is the starting point of curve $\mathcal{C}$ and $\vec{r}(t_b)$ is its end point. These may be the same point even if $t_a \neq t_b$ (*e.g.* integral once around a circle from $\theta = 0$ to $\theta = 2\pi$).
Likewise, we can use the *limit-of-a-sum* definition to make sense of many other types of line integrals such as

$$\int_{\mathcal{C}} f(\vec{r})\,|d\vec{r}|, \quad \int_{\mathcal{C}} \vec{F}\,|d\vec{r}|, \quad \int_{\mathcal{C}} f(\vec{r})\,d\vec{r}, \quad \int_{\mathcal{C}} \vec{F}\times d\vec{r}.$$

The first one gives a scalar result and the latter three give *vector* results.
▶ One important example is

$$\int_{\mathcal{C}} |d\vec{r}| = \lim_{|\Delta\vec{r}_n|\to 0} \sum_{n=1}^{N} |\Delta\vec{r}_n| \tag{4}$$

which is the *length* of the curve $\mathcal{C}$ from its starting point $\vec{r}_a = \vec{r}_0$ to its end point $\vec{r}_b = \vec{r}_N$. If a parametrization $\vec{r} = \vec{r}(t)$ is known then

$$\int_{\mathcal{C}} |d\vec{r}| = \int_{t_a}^{t_b} \left|\frac{d\vec{r}(t)}{dt}\right| |dt| \tag{5}$$

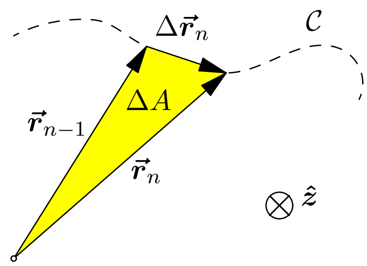
where $\vec{\boldsymbol{r}}_a = \vec{\boldsymbol{r}}(t_a)$ and $\vec{\boldsymbol{r}}_b = \vec{\boldsymbol{r}}(t_b)$. That's *almost* a 221 integral, except for that $|dt|$, what does that mean?! Again you can understand that from the limit-of-a-sum definition with $t_0 = t_a$, $t_N = t_b$ and $\Delta t_n = t_n - t_{n-1}$. If $t_a < t_b$ then $\Delta t_n > 0$ and $dt > 0$, so $|dt| = dt$ and we're blissfully happy. But if $t_b < t_a$ then $\Delta t_n < 0$ and $dt < 0$, so $|dt| = -dt$ and

$$\int_{t_a}^{t_b} (\cdots)|dt| = \int_{t_b}^{t_a} (\cdots)dt, \qquad \text{if} \quad t_a > t_b. \tag{6}$$

► A special example of a $\int_{\mathcal{C}} \vec{\boldsymbol{F}} \times d\vec{\boldsymbol{r}}$ integral is

$$\int_{\mathcal{C}} \vec{\boldsymbol{r}} \times d\vec{\boldsymbol{r}} = \lim_{|\Delta\vec{\boldsymbol{r}}_n|\to 0} \sum_{n=1}^{N} \vec{\boldsymbol{r}}_n \times \Delta\vec{\boldsymbol{r}}_n = \int_{t_a}^{t_b} \left(\vec{\boldsymbol{r}}(t) \times \frac{d\vec{\boldsymbol{r}}(t)}{dt} \right) dt \tag{7}$$

This integral yields a *vector* $2A\hat{\boldsymbol{z}}$ whose magnitude is *twice* the area A swept by the position vector $\vec{\boldsymbol{r}}(t)$ *if and only if* the curve $\mathcal{C}$ lies in a plane perpendicular to $\hat{\boldsymbol{z}}$ and O is in that plane (recall Kepler's law that *'the radius vector sweeps equal areas in equal times'*). This follows directly from the fact that $\vec{\boldsymbol{r}}_n \times \Delta\vec{\boldsymbol{r}}_n = 2(\Delta A)\hat{\boldsymbol{z}}$ is the area of the parallelogram with sides $\vec{\boldsymbol{r}}_n$ and $\Delta\vec{\boldsymbol{r}}_n$ which is twice the area ΔA of the triangle $\vec{\boldsymbol{r}}_{n-1}$, $\Delta\vec{\boldsymbol{r}}_n$, $\vec{\boldsymbol{r}}_n$. If $\mathcal{C}$ and O are not coplanar then the vectors $\vec{\boldsymbol{r}}_n \times \Delta\vec{\boldsymbol{r}}_n$ are not all in the same direction and their *vector* sum is *not* the area swept by $\vec{\boldsymbol{r}}$. In that more general case, the surface is conical and to calculate its area S we would need to calculate $S = \frac{1}{2}\int_{\mathcal{C}} |\vec{\boldsymbol{r}} \times d\vec{\boldsymbol{r}}|$.



Exercises:

1. What is the curve described by $\vec{\boldsymbol{r}}(t) = a\cos\omega t\,\hat{\boldsymbol{x}} + a\sin\omega t\,\hat{\boldsymbol{y}} + bt\,\hat{\boldsymbol{z}}$, where a, b and ω are constant real numbers and $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ represent the orthonormal unit vectors for cartesian coordinates? What is the tangent to the curve at $\vec{\boldsymbol{r}}(t)$?

2. Consider the vector function $\vec{\boldsymbol{r}}(\theta) = \vec{\boldsymbol{r}}_c + a\cos\theta\,\vec{\boldsymbol{e}}_1 + b\sin\theta\,\vec{\boldsymbol{e}}_2$, where $\vec{\boldsymbol{r}}_c$, $\vec{\boldsymbol{e}}_1$, $\vec{\boldsymbol{e}}_2$, a and b are constants, with $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = \delta_{ij}$. What kind of curve is this? Next, assume that $\vec{\boldsymbol{r}}_c$, $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$ are in the same plane. Consider cartesian coordinates (x, y) in that plane such that $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$. Assume that the angle between $\vec{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{x}}$ is α. Derive the equation of the curve in terms of the cartesian coordinates (x, y) (i) in parametric form, (ii) in implicit form $f(x, y) = 0$. Simplify your equations as much as possible. Find a geometric interpretation for the parameter θ.

3. Generalize the previous exercise to the case where $\vec{\boldsymbol{r}}_c$ is not in the same plane as $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$. Consider general cartesian coordinates (x, y, z) such that $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Assume that all the angles between $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$ and the basis vectors $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ are known. How many independent angles is that? Specify those angles. Derive the parametric equations of the curve for the cartesian coordinates (x, y, z) in terms of the parameter θ.

4. Derive integrals for the length and area of the planar curve in the previous exercise. Clean up your integrals and compute them if possible.

5. Calculate $\int_{\mathcal{C}} d\vec{\boldsymbol{r}}$ and $\int_{\mathcal{C}} \vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}}$ along the curve of the preceding exercise from $\vec{\boldsymbol{r}}(0)$ to $\vec{\boldsymbol{r}}(-3\pi/2)$.

6. Calculate $\int_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ and $\int_{\mathcal{C}} \vec{\boldsymbol{F}} \times d\vec{\boldsymbol{r}}$ with $\vec{\boldsymbol{F}} = (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}})/|\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}}|^2$ when $\mathcal{C}$ is the circle of radius R in the x, y plane centered at the origin. How do the integrals depend on R?

1.3 Surfaces

Recall the parametric equation of a plane: $\vec{\boldsymbol{r}}(u,v) = \vec{\boldsymbol{r}}_0 + u\,\vec{\boldsymbol{a}} + v\,\vec{\boldsymbol{b}}$ where $\vec{\boldsymbol{r}}_0$ is a point on the plane, $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{b}}$ are two vectors parallel to the plane (but not to each other) and u, v are real parameters. These parameters specify *coordinates* for the surface. More generally, the parametric equation of a surface prescribes the position vector $\vec{\boldsymbol{r}}$ as a function of two real parameters, u and v say, $\vec{\boldsymbol{r}}(u,v)$. Again, the names of the parameters do not matter, $\vec{\boldsymbol{r}}(s,t)$ is also common. The following two examples are fundamental.

▶ If the surface can be parametrized by the cartesian coordinates x and y, *i.e.* the surface is described by $z = h(x,y)$, then the position vector of a point on the surface is

$$\vec{\boldsymbol{r}}(x,y) = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + h(x,y)\,\hat{\boldsymbol{z}}, \tag{8}$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are the unit vectors in the respective coordinate directions.

▶ The surface of a sphere of radius R centered at $\vec{\boldsymbol{r}}_c$ can be parametrized by

$$\vec{\boldsymbol{r}}(u,v) = \vec{\boldsymbol{r}}_c + \vec{\boldsymbol{e}}_1 R\cos u\cos v + \vec{\boldsymbol{e}}_2 R\sin u\cos v + \vec{\boldsymbol{e}}_3 R\sin v, \tag{9}$$

where $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = \delta_{ij}$. Verify that $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_c| = R$ for any real u and v. The parameters u and v can be interpreted as angles. With $\vec{\boldsymbol{e}}_3$ as the polar axis, v is the latitude and u is the longitude. The parametric representation $\vec{\boldsymbol{r}}(\varphi,\theta) = \vec{\boldsymbol{r}}_c + \vec{\boldsymbol{e}}_1 R\cos\varphi\sin\theta + \vec{\boldsymbol{e}}_2 R\sin\varphi\sin\theta + \vec{\boldsymbol{e}}_3 R\cos\theta$, is a different but equally valid parametric representation of the same sphere. Here θ is the polar angle, or co-latitude. It is the angle between the position vector and the polar axis $\vec{\boldsymbol{e}}_3$. The angle φ is the azimuthal, or longitudinal, angle.

Coordinate curves and Tangent vectors If one of the parameters is held fixed, $v = v_0$ say, we obtain a curve $\vec{\boldsymbol{r}}(u,v_0)$. There is one such curve for every value of v. For the sphere parametrized as in (9), $\vec{\boldsymbol{r}}(u,v_0)$ is the *v_0-parallel*, the circle at latitude v_0. Likewise $\vec{\boldsymbol{r}}(u_0,v)$ describes another curve. This would be a longitude circle, or *meridian*, for the sphere. The set of all such curves *generates* the surface. These two families of curves are *parametric curves* or *coordinate curves* for the surface. The vectors $\partial\vec{\boldsymbol{r}}/\partial u$ and $\partial\vec{\boldsymbol{r}}/\partial v$ are tangent to their respective parametric curves and hence to the surface. These two vectors taken at the same point $\vec{\boldsymbol{r}}(u,v)$ define the tangent plane at that point. The coordinates are said to be *orthogonal* if the tangent vectors $\partial\vec{\boldsymbol{r}}/\partial u$ and $\partial\vec{\boldsymbol{r}}/\partial v$ at each point $\vec{\boldsymbol{r}}(u,v)$ are orthogonal to each other.

Normal to the surface at a point At any point $\vec{\boldsymbol{r}}(u,v)$ on a surface, there is an infinity of tangent directions but there is only one normal direction. The normal to the surface at a point $\vec{\boldsymbol{r}}(u,v)$ is given by

$$\vec{\boldsymbol{N}} = \frac{\partial\vec{\boldsymbol{r}}}{\partial u} \times \frac{\partial\vec{\boldsymbol{r}}}{\partial v}. \tag{10}$$

Note that the ordering (u,v) specifies an orientation for the surface, *i.e.* an 'up' and 'down' side.

Surface element: The surface element $d\vec{\boldsymbol{S}}$ at a point $\vec{\boldsymbol{r}}$ on a surface S is a vector perpendicular to the surface at that point with a magnitude dS that is the area of an 'infinitesimal' patch of surface at that point. Just like in the case of the line element along a curve, this is a useful shortcut for the *limit-of-a-sum* interpretation of integrals. The surface element is often written $d\vec{\boldsymbol{S}} = \hat{\boldsymbol{n}}dS$ where $\hat{\boldsymbol{n}}$ is the unit normal to the surface at that point. If a parametric representation $\vec{\boldsymbol{r}}(u,v)$ for the surface is known then

$$d\vec{\boldsymbol{S}} = \left(\frac{\partial\vec{\boldsymbol{r}}}{\partial u} \times \frac{\partial\vec{\boldsymbol{r}}}{\partial v}\right) du dv, \tag{11}$$

since the right hand side represents the area of the parallelogram formed by the line elements $\frac{\partial \vec{\boldsymbol{r}}}{\partial u}du$ and $\frac{\partial \vec{\boldsymbol{r}}}{\partial v}dv$. Note that $\hat{\boldsymbol{n}} \neq \vec{\boldsymbol{N}}$ but $\hat{\boldsymbol{n}} = \vec{\boldsymbol{N}}/|\vec{\boldsymbol{N}}|$ since $\hat{\boldsymbol{n}}$ is a unit vector. Although we often need to refer to the unit normal $\hat{\boldsymbol{n}}$, it is usually not needed to compute it explicitly since in practice it is the area element $d\vec{\boldsymbol{S}} = \vec{\boldsymbol{N}}dudv$ which is required.

Exercises:

1. Compute tangent vectors and the normal to the surface $z = h(x, y)$. Show that $\partial\vec{\boldsymbol{r}}/\partial x$ and $\partial\vec{\boldsymbol{r}}/\partial y$ are not orthogonal to each other in general. Determine the class of functions $h(x, y)$ for which (x, y) are orthogonal coordinates on the surface $z = h(x, y)$ and interpret geometrically. Derive an explicit formula for the area element $dS = |d\vec{\boldsymbol{S}}|$ in terms of $h(x, y)$.

2. Deduce from the implicit equation $|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_c| = R$ for a sphere of radius R and center $\vec{\boldsymbol{r}}_c$ that $(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_c)\cdot\partial\vec{\boldsymbol{r}}/\partial u = (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_c)\cdot\partial\vec{\boldsymbol{r}}/\partial v = 0$ for any u and v, where $\vec{\boldsymbol{r}}(u, v)$ is *any* parametrization of a point on the sphere. Compute $\partial\vec{\boldsymbol{r}}/\partial u$ and $\partial\vec{\boldsymbol{r}}/\partial v$ for the parametrization (9) and verify that these vectors are perpendicular to the radial vector $\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_c$ and to each other (when evaluated at the same point on the sphere, whatever that point is). Orthogonality of $\partial\vec{\boldsymbol{r}}/\partial u$ and $\partial\vec{\boldsymbol{r}}/\partial v$ implies that those u and v are orthogonal coordinates for the sphere. Compute the surface element $d\vec{\boldsymbol{S}}$ for the sphere in terms of both the longitude-latitude parametrization (9) and the longitude-colatitude parametrization φ, θ. Do the surface elements point toward or away from the center of the sphere?

3. Explain why the surface described by $\vec{\boldsymbol{r}}(u, v) = \hat{\boldsymbol{x}}\, a \cos u \cos v + \hat{\boldsymbol{y}}\, b \sin u \cos v + \hat{\boldsymbol{z}}\, c \sin v$ where a, b and c are real constants is the surface of an *ellipsoid*. Are u and v orthogonal coordinates for that surface? Consider cartesian coordinates (x, y, z) such that $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Derive the implicit equation $f(x, y, z) = 0$ satisfied by all such $\vec{\boldsymbol{r}}(u, v)$'s.

4. Consider the vector function $\vec{\boldsymbol{r}}(u, v) = \vec{\boldsymbol{r}}_c + \vec{\boldsymbol{e}}_1\, a \cos u \cos v + \vec{\boldsymbol{e}}_2\, b \sin u \cos v + \vec{\boldsymbol{e}}_3\, c \sin v$, where $\vec{\boldsymbol{r}}_c, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3$, a, b and c are constants with $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = \delta_{ij}$. What does the set of all such $\vec{\boldsymbol{r}}(u, v)$'s represent? Consider cartesian coordinates (x, y, z) such that $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Assume that all the angles between $\vec{\boldsymbol{e}}_1$, $\vec{\boldsymbol{e}}_2$, $\vec{\boldsymbol{e}}_3$ and the basis vectors $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ are known. How many independent angles is that? Specify which angles. Can you assume $\{\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3\}$ is right handed? Explain. Derive the implicit equation $f(x, y, z) = 0$ satisfied by all such $\vec{\boldsymbol{r}}(u, v)$'s. Express your answer in terms of the minimum independent angles that you specified earlier.

5. Explain why the surface of a torus (*i.e.* 'donut' or tire) can be parametrized as $x = (R + a\cos\theta)\cos\varphi$, $y = (R + a\cos\theta)\sin\varphi$, $z = a\sin\theta$. Interpret the geometric meaning of the parameters R, a, φ and θ. What are the ranges of φ and θ needed to cover the entire torus? Do these parameters provide orthogonal coordinates for the torus? Calculate the surface element $d\vec{\boldsymbol{S}}$.

1.4 Surface integrals

The typical surface integral is of the form $\int_S \vec{\boldsymbol{v}}\cdot d\vec{\boldsymbol{S}}$. This represents the *flux* of $\vec{\boldsymbol{v}}$ through the surface S. If $\vec{\boldsymbol{v}}(\vec{\boldsymbol{r}})$ is the velocity of a fluid, water or air, at point $\vec{\boldsymbol{r}}$, then $\int_S \vec{\boldsymbol{v}}\cdot d\vec{\boldsymbol{S}}$ is the time-rate at which volume of fluid is flowing through that surface per unit time. Indeed, $\vec{\boldsymbol{v}}\cdot d\vec{\boldsymbol{S}} = (\vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}})\, dS$ where $\hat{\boldsymbol{n}}$ is the unit normal to the surface and $\vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}$ is the component of fluid velocity that is perpendicular to the surface. If that component is zero, the fluid moves tangentially to the surface, not through the surface. Speed $\times$ area $=$ volume per unit time, so $\vec{\boldsymbol{v}}\cdot d\vec{\boldsymbol{S}}$ is the volume of fluid passing through

the surface element $d\vec{S}$ per unit time at that point at that time. The total volume passing through the entire surface S per unit time is $\int_S \vec{v} \cdot d\vec{S}$. Such integrals are often called *flux* integrals.
We can make sense of such integrals as the *limit of a sum*, for instance by picking a series of points on the surface, then forming a triangular meshing of the surface and evaluating the integral as the limit of the sum of $\vec{v}$ evaluated at the center of such triangles dotted with the unit normal to the triangle and times the area of the triangle (making sure of course that for each triangle we pick its normal $\hat{n}$ to correspond to the same side of the surface as the neighboring elements).
If a parametric representation $\vec{r}(u, v)$ is known for the surface then we can also write

$$\int_S \vec{v} \cdot d\vec{S} = \int_A \left(\vec{v} \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) \right) du dv, \tag{12}$$

where A is the domain in the u, v parameter plane that corresponds to S.
As for line integrals, we can make sense of many other types of surface integrals such as

$$\int_S p \, d\vec{S},$$

which would represent the net pressure force on S if $p = p(\vec{r})$ is the pressure at point $\vec{r}$. Other surface integrals could have the form $\int_S \vec{v} \times d\vec{S}$, etc. In particular the total area of surface S is $\int_S |d\vec{S}|$.

Exercises:

1. Compute the area of the sphere and the area of a spherical cap (*e.g.* area of arctic cap).
2. Provide an explicit integral for the total surface area of the torus of outer radius R and inner radius a.
3. Calculate $\int_S \vec{r} \cdot d\vec{S}$ where S is (i) the square $0 \le x, y \le a$ at $z = b$, (ii) the surface of the sphere of radius R centered at $(0, 0, 0)$, (iii) the surface of the sphere of radius R centered at $x = x_0$, $y = z = 0$.
4. Calculate $\int_S (\vec{r}/r^3) \cdot d\vec{S}$ where S is the surface of the sphere of radius R centered at the origin. How does the result depend on R?
5. The pressure *outside* the sphere of radius R centered at $\vec{r}_c$ is $p = p_0 + A\vec{r} \cdot \vec{e}$ where $\vec{e}$ is a fixed unit vector and p_0 and A are constants. The pressure inside the sphere is the constant $p_1 > p_0$. Calculate the net force on the sphere. Calculate the net torque on the sphere about its center $\vec{r}_c$ and about the origin.

1.5 Volumes and volume integrals

We have seen that $\vec{r}(t)$ is the parametric equation of a curve, $\vec{r}(u, v)$ represents a surface, now we discuss $\vec{r}(u, v, w)$ which is the parametric representation of a volume. Curves $\vec{r}(t)$ are one dimensional objects so they have only one parameter t or each point on the known curve is determined by a single *coordinate*. Surfaces are two-dimensional and require two parameters u and v, which are coordinates for points on that surface. Volumes are three-dimensional objects that require three parameters u, v, w say. Each point is specified by three *coordinates*.
► A sphere of radius R centered at $\vec{r}_c$ has the implicit equation $|\vec{r} - \vec{r}_c| \le R$ or $(\vec{r} - \vec{r}_c) \cdot (\vec{r} - \vec{r}_c) \le R^2$ to avoid square roots. In cartesian coordinates this translates into the implicit equation

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 \le R^2. \tag{13}$$

An explicit parametrization for that sphere is

$$\vec{\boldsymbol{r}}(u,v,w) = \vec{\boldsymbol{r}}_c + \vec{\boldsymbol{e}}_1\, w\cos u\cos v + \vec{\boldsymbol{e}}_2\, w\sin u\cos v + \vec{\boldsymbol{e}}_3\, w\sin v, \tag{14}$$

or

$$\vec{\boldsymbol{r}}(r,\theta,\varphi) = \vec{\boldsymbol{r}}_c + \vec{\boldsymbol{e}}_1\, r\sin\theta\cos\varphi + \vec{\boldsymbol{e}}_2\, r\sin\theta\sin\varphi + \vec{\boldsymbol{e}}_3\, r\cos\theta, \tag{15}$$

where $\{\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3\}$ are *any* three orthonormal vectors such that $\vec{\boldsymbol{e}}_i \cdot \vec{\boldsymbol{e}}_j = \delta_{ij}$. For the parametrization (14) we recognize u and v as the longitude and latitude parameters used earlier for spherical surfaces (9) and note that w has units of length. From orthonormality of $\{\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3\}$ we deduce that $(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_c)\cdot(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_c) = w^2$, so w can be interpreted as the distance from the center of the sphere. To parametrize the entire sphere we need to consider all the u, v and w's such that

$$0 \le u < 2\pi, \qquad -\frac{\pi}{2} \le v \le \frac{\pi}{2}, \qquad 0 \le w \le R. \tag{16}$$

The parametrization (15) is mathematically equivalent to this u, v, w parametrization but it is in the standard form of *spherical coordinates*, with r representing the distance to the origin, φ the azimuthal (or longitude) angle and θ the polar angle. To describe the full sphere we need

$$0 \le r \le R, \qquad 0 \le \varphi < 2\pi, \qquad 0 \le \theta \le \pi. \tag{17}$$

Coordinate curves

For a curve $\vec{\boldsymbol{r}}(t)$, all we needed to worry about was the tangent $d\vec{\boldsymbol{r}}/dt$ and the line element $d\vec{\boldsymbol{r}} = (d\vec{\boldsymbol{r}}/dt)dt$. For surfaces, $\vec{\boldsymbol{r}}(u,v)$ we have two sets of coordinates curves with tangents $\partial\vec{\boldsymbol{r}}/\partial u$ and $\partial\vec{\boldsymbol{r}}/\partial v$, a normal $\vec{\boldsymbol{N}} = (\partial\vec{\boldsymbol{r}}/\partial u)\times(\partial\vec{\boldsymbol{r}}/\partial v)$ and a surface element $d\vec{\boldsymbol{S}} = \vec{\boldsymbol{N}}dudv$. Now for volumes $\vec{\boldsymbol{r}}(u,v,w)$, we have three sets of coordinates curves with tangents $\partial\vec{\boldsymbol{r}}/\partial u$, $\partial\vec{\boldsymbol{r}}/\partial v$ and $\partial\vec{\boldsymbol{r}}/\partial w$. A u-coordinate curve for instance, corresponds to $\vec{\boldsymbol{r}}(u,v,w)$ with v and w fixed. There is a double infinity of such one dimensional curves, one for each v, w pair. For the parametrization (14), the u-coordinate curves correspond to *parallels*, *i.e.* circles of fixed radius w at fixed latitude v. The v-coordinate curves are *meridians*, *i.e.* circles of fixed radius w through the poles. The w-coordinate curves are radial lines out of the origin.

Coordinate surfaces

For volumes $\vec{\boldsymbol{r}}(u,v,w)$, we also have three sets of *coordinate surfaces* corresponding to one parameter fixed and the other two free. A w-isosurface for instance corresponds to $\vec{\boldsymbol{r}}(u,v,w)$ for a fixed w. There is a single infinity of such two dimensional (u,v) surfaces. For the parametrization (14) such surfaces correspond to spherical surfaces of radius w centered at $\vec{\boldsymbol{r}}_c$. Likewise, if we fix u but let v and w free, we get another surface, and v fixed with u and w free is another coordinate surface.

Line Elements

Thus given a volume parametrization $\vec{\boldsymbol{r}}(u,v,w)$ we can define four types of line elements, one for each of the coordinate directions $(\partial\vec{\boldsymbol{r}}/\partial u)du$, $(\partial\vec{\boldsymbol{r}}/\partial v)dv$, $(\partial\vec{\boldsymbol{r}}/\partial w)dw$ and a general line element corresponding to the infinitesimal displacement from coordinates (u,v,w) to the coordinates $(u + du, v + dv, w + dw)$. That general line element $d\vec{\boldsymbol{r}}$ is given by (chain rule):

$$d\vec{\boldsymbol{r}} = \frac{\partial\vec{\boldsymbol{r}}}{\partial u}du + \frac{\partial\vec{\boldsymbol{r}}}{\partial v}dv + \frac{\partial\vec{\boldsymbol{r}}}{\partial w}dw. \tag{18}$$

Surface Elements

Likewise, there are three basic types of *surface elements*, one for each coordinate surface. The surface element on a w-isosurface, for example, is given by

$$d\vec{\boldsymbol{S}}_w = \left(\frac{\partial\vec{\boldsymbol{r}}}{\partial u} \times \frac{\partial\vec{\boldsymbol{r}}}{\partial v}\right) dudv, \tag{19}$$

while the surface elements on a u-isosurface and a v-isosurface are respectively

$$d\vec{\boldsymbol{S}}_u = \left(\frac{\partial \vec{\boldsymbol{r}}}{\partial v} \times \frac{\partial \vec{\boldsymbol{r}}}{\partial w}\right) dv dw, \qquad d\vec{\boldsymbol{S}}_v = \left(\frac{\partial \vec{\boldsymbol{r}}}{\partial w} \times \frac{\partial \vec{\boldsymbol{r}}}{\partial u}\right) du dw. \tag{20}$$

Note that surface orientations are built into the order of the coordinates.

Volume Element

Last but not least, a parametrization $\vec{\boldsymbol{r}}(u, v, w)$ defines a *volume element* given by the mixed (i.e. triple scalar) product

$$dV = \left(\frac{\partial \vec{\boldsymbol{r}}}{\partial u} \times \frac{\partial \vec{\boldsymbol{r}}}{\partial v}\right) \cdot \frac{\partial \vec{\boldsymbol{r}}}{\partial w}\, du\, dv\, dw \equiv \det\left(\frac{\partial \vec{\boldsymbol{r}}}{\partial u}, \frac{\partial \vec{\boldsymbol{r}}}{\partial v}, \frac{\partial \vec{\boldsymbol{r}}}{\partial w}\right) du\, dv\, dw. \tag{21}$$

The definition of volume integrals as limit-of-a-sum should be obvious by now. If an explicit parametrization $\vec{\boldsymbol{r}}(u, v, w)$ for the volume is known, we can use the volume element (21) and write the volume integral in $\vec{\boldsymbol{r}}$ space as an iterated triple integral over u, v, w. Be careful that there is an orientation implicitly built into the ordering of the parameters, as should be obvious from the definition of the mixed product and determinants. The volume element dV is usually meant to be positive so the sign of the mixed product and the bounds of integrations for the parameters u, v and w must be chosen to respect that. (Recall the definition of $|dt|$ in the line integrals section).

Exercises

1. Calculate the line, surface and volume elements for the coordinates (14). You need to calculate 4 line elements and 3 surfaces elements. One line element for each coordinate curve and the general line element. Verify that these coordinates are orthogonal.

2. Formulate integral expressions in terms of the coordinates (14) and (15) for the surface and volume of a sphere of radius R. Calculate those integrals.

3. A curve $\vec{\boldsymbol{r}}(t)$ is given in terms of the (u, v, w) coordinates (14), *i.e.* $\vec{\boldsymbol{r}}(t) = \vec{\boldsymbol{r}}(u, v, w)$ with $(u(t), v(t), w(t))$ for $t = t_a$ to $t = t_b$. Find an explicit expression in terms of $(u(t), v(t), w(t))$ as a t-integral for the length of that curve.

4. Find suitable coordinates for a torus. Are your coordinates orthogonal? Compute the volume of that torus.

1.6 Mappings, curvilinear coordinates

Parametrizations of curves, surfaces and volumes is essentially equivalent to the concepts of 'mappings' and curvilinear coordinates.

Mappings

The parametrization (9) for the surface of a sphere of radius R provides a *mapping* of that surface to the $0 \le u < 2\pi$, $-\pi/2 \le v \le \pi/2$ rectangle in the (u, v) plane. In a mapping $\vec{\boldsymbol{r}}(u, v)$ a small rectangle of sides du, dv at a point (u, v) in the (u, v) plane is mapped to a small parallelogram of sides $(\partial\vec{\boldsymbol{r}}/\partial u)du, (\partial\vec{\boldsymbol{r}}/\partial v)dv$ at point $\vec{\boldsymbol{r}}(u, v)$ in the Euclidean space.

The parametrization (14) for the sphere of radius R provides a *mapping* from the sphere of radius R in Euclidean space to the box $0 \le u < 2\pi$, $-\pi/2 \le v \le \pi/2$, $0 \le w \le R$ in the (u, v, w) space. In a mapping $\vec{\boldsymbol{r}}(u, v, w)$, the infinitesimal box of sides du, dv, dw located at point (u, v, w) in the (u, v, w) space is mapped to a parallelepiped of sides $(\partial\vec{\boldsymbol{r}}/\partial u)du, (\partial\vec{\boldsymbol{r}}/\partial v)dv, (\partial\vec{\boldsymbol{r}}/\partial w)dw$ at the point $\vec{\boldsymbol{r}}(u, v, w)$ in the Euclidean space.

Curvilinear coordinates, orthogonal coordinates

The parametrizations $\vec{r}(u,v)$ and $\vec{r}(u,v,w)$ define *coordinates* for a surface or a volume, respectively. If the coordinate curves are not straight lines one talks of *curvilinear coordinates.* These mappings define good coordinates if the coordinate curves intersect *transversally i.e.* if the coordinate curves are not tangent to each other. If the coordinate curves intersect transversally at a point then the coordinates tangent vectors at that point provide linearly independent directions in the space of $\vec{r}$. Tangent intersections at a point $\vec{r}$ would imply that the tangent vectors are not linearly independent at that point and that $(\partial\vec{r}/\partial u)\times(\partial\vec{r}/\partial v)=0$ at $\vec{r}(u,v)$ in the surface case or that $\det(\partial\vec{r}/\partial u,\partial\vec{r}/\partial v,\partial\vec{r}/\partial w)=0$ at that point $\vec{r}(u,v,w)$ in the volume case.
The coordinates (u,v,w) are *orthogonal* if the coordinate curves in $\vec{r}$-space intersect at right angles. This is the best kind of 'transversal' intersection and these are the most desirable type of coordinates, however non-orthogonal coordinates are sometimes more convenient for some problems. Two fundamental examples of orthogonal curvilinear coordinates are

► *Cylindrical (or polar) coordinates*

$$x=\rho\cos\theta,\quad y=\rho\sin\theta,\quad z=z. \tag{22}$$

► *Spherical coordinates*

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta. \tag{23}$$

Changing notation from (x,y,z) to (x_1,x_2,x_3) and from (u,v,w) to (q_1,q_2,q_3) a general change of coordinates from cartesian $(x,y,z)\equiv(x_1,x_2,x_3)$ to curvilinear (q_1,q_2,q_3) coordinates is expressed succinctly by

$$x_i=x_i(q_1,q_2,q_3),\quad i=1,2,3. \tag{24}$$

The position vector $\vec{r}$ can be expressed in terms of the q_j's through the cartesian expression:

$$\vec{r}(q_1,q_2,q_3)=\hat{\boldsymbol{x}}\,x(q_1,q_2,q_3)+\hat{\boldsymbol{y}}\,y(q_1,q_2,q_3)+\hat{\boldsymbol{z}}\,z(q_1,q_2,q_3)=\sum_{i=1}^{3}\vec{e}_i\,x_i(q_1,q_2,q_3), \tag{25}$$

where $\vec{e}_i\cdot\vec{e}_j=\delta_{ij}$. The q_i coordinate curve is the curve $\vec{r}(q_1,q_2,q_3)$ where q_i is free but the other two variables are fixed. The q_i isosurface is the surface $\vec{r}(q_1,q_2,q_3)$ where q_i is fixed and the other two parameters are free.
The coordinate tangent vectors $\partial\vec{r}/\partial q_i$ are key to the coordinates. They provide a natural vector basis for those coordinates. The coordinates are orthogonal if these tangent vectors are orthogonal to each other at each point. In that case it is useful to define the unit vector $\hat{\boldsymbol{q}}_i$ in the q_i coordinate direction by

$$h_i=\left|\frac{\partial\vec{r}}{\partial q_i}\right|,\qquad \frac{\partial\vec{r}}{\partial q_i}=h_i\hat{\boldsymbol{q}}_i, \tag{26}$$

where h_i is the the magnitude of the tangent vector in the q_i direction, $\partial\vec{r}/\partial q_i$, and $\hat{\boldsymbol{q}}_i\cdot\hat{\boldsymbol{q}}_j=\delta_{ij}$ for orthogonal coordinates. These h_i's are called the *scale* factors. The distance traveled in x-space when changing q_i by dq_i, keeping the other q's fixed, is $|d\vec{r}|=h_i dq_i$ (no summation). The distance ds travelled in x-space when the orthogonal curvilinear coordinates change from (q_1,q_2,q_3) to $(q_1+dq_1,q_2+dq_2,q_3+dq_3)$ is

$$ds^2=d\vec{r}\cdot d\vec{r}=h_1^2\,dq_1^2+h_2^2\,dq_2^2+h_3^2\,dq_3^2. \tag{27}$$

Although the cartesian unit vectors $\vec{\boldsymbol{e}}_i$ are independent of the coordinates, the curvilinear unit vectors $\hat{\boldsymbol{q}}_i$ in general *are* functions of the coordinates, even if the latter are orthogonal. Hence $\partial\hat{\boldsymbol{q}}_i/\partial q_j$ is in general non-zero. For orthogonal coordinates, those derivatives $\partial\hat{\boldsymbol{q}}_i/\partial q_j$ can be expressed in terms of the scale factors and the unit vectors.

For instance, for spherical coordinates $(q_1, q_2, q_3) \equiv (r, \varphi, \theta)$, the unit vector in the $q_1 \equiv r$ direction is the vector

$$\frac{\partial\vec{\boldsymbol{r}}(r,\varphi,\theta)}{\partial r} = \hat{\boldsymbol{x}}\,\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\,\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\,\cos\theta, \tag{28}$$

so the scale coefficient $h_1 \equiv h_r = 1$ and the unit vector

$$\hat{\boldsymbol{q}}_1 \equiv \hat{\boldsymbol{r}} \equiv \vec{\boldsymbol{e}}_r = \hat{\boldsymbol{x}}\,\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\,\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\,\cos\theta. \tag{29}$$

The position vector $\vec{\boldsymbol{r}}$ can be expressed as

$$\boxed{\vec{\boldsymbol{r}} = r\,\hat{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \;\equiv\; x_i\,\vec{\boldsymbol{e}}_i} \tag{30}$$

with *summation convention* in that last expression. So its expression in terms of spherical coordinates and their unit vectors, $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}$, is simpler than in cartesian coordinates, $\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, but there is a catch! The radial unit vector $\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}(\varphi,\theta)$ varies in the azimuthal and polar angle directions, while the cartesian unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are independent of the coordinates!

For orthogonal coordinates, the scale factors h_i's determine everything. In particular, the surface and volume elements can be expressed in terms of the h_i's. For instance, the surface element for a q_3-isosurface is

$$d\vec{\boldsymbol{S}}_3 = \hat{\boldsymbol{q}}_3\,h_1h_2\,dq_1dq_2, \tag{31}$$

and the volume element

$$dV = h_1h_2h_3\,dq_1dq_2dq_3, \tag{32}$$

assuming that q_1, q_2, q_3 is right-handed. These follow directly from (19) and (21) and (26) when the coordinates are orthogonal.

Exercises

1. Find the scale factors h_i and the unit vectors $\hat{\boldsymbol{q}}_i$ for cylindrical and spherical coordinates. Express the 3 surface elements and the volume element in terms of those scale factors and unit vectors. Sketch the unit vector $\hat{\boldsymbol{q}}_i$ in the (x, y, z) space (use several 'views' rather than trying to make an ugly 3D sketch!). Express the position vector $\vec{\boldsymbol{r}}$ in terms of the unit vectors $\hat{\boldsymbol{q}}_i$. Calculate the derivatives $\partial\hat{\boldsymbol{q}}_i/\partial q_j$ for all i, j and express these derivatives in terms of the scale factors h_k and the unit vectors $\hat{\boldsymbol{q}}_k$, $k = 1, 2, 3$.

2. A curve in the (x, y) plane is given in terms of polar coordinates as $\rho = \rho(\theta)$. Deduce θ-integral expressions for the length of the curve and for the area swept by the radial vector.

3. Consider *elliptical coordinates* (u, v, w) defined by $x = \alpha\cosh u\cos v$, $y = \alpha\sinh u\sin v$, $z = w$ for some $\alpha > 0$, where x, y and z are standard cartesian coordinates in 3D Euclidean space. What do the coordinate curves correspond to in the (x, y, z) space? Are these orthogonal coordinates? What is the volume element in terms of elliptical coordinates?

4. For general curvilinear coordinates, not necessarily orthogonal, is the q_i-isosurface perpendicular to $\partial\vec{\boldsymbol{r}}/\partial q_i$? is it orthogonal to $(\partial\vec{\boldsymbol{r}}/\partial q_j) \times (\partial\vec{\boldsymbol{r}}/\partial q_k)$ where i, j, k are all distinct? What about for orthogonal coordinates?

1.7 Change of variables

Parametrizations of surfaces and volumes and curvilinear coordinates are geometric examples of a *change of variables*. These change of variables and the associated formula and geometric concepts can occur in other non-geometric contexts. The fundamental relationship is the formula for a volume element (21). In the context of a general change of variables from (x_1, x_2, x_3) to (q_1, q_2, q_3) that formula (21) reads

$$dx_1 dx_2 dx_3 = dV_x = J\, dq_1 dq_2 dq_3 = J\, dV_q \tag{33}$$

where

$$J = \begin{vmatrix} \partial x_1/\partial q_1 & \partial x_1/\partial q_2 & \partial x_1/\partial q_3 \\ \partial x_2/\partial q_1 & \partial x_2/\partial q_2 & \partial x_2/\partial q_3 \\ \partial x_3/\partial q_1 & \partial x_3/\partial q_2 & \partial x_3/\partial q_3 \end{vmatrix} = \det\left(\frac{\partial x_i}{\partial q_j}\right) \tag{34}$$

is the *Jacobian determinant* and dV_x is a volume element in the x-space while dV_q is the corresponding volume element in q-space. The Jacobian is the determinant of the Jacobian *matrix*

$$\boldsymbol{J} = \begin{bmatrix} \partial x_1/\partial q_1 & \partial x_1/\partial q_2 & \partial x_1/\partial q_3 \\ \partial x_2/\partial q_1 & \partial x_2/\partial q_2 & \partial x_2/\partial q_3 \\ \partial x_3/\partial q_1 & \partial x_3/\partial q_2 & \partial x_3/\partial q_3 \end{bmatrix} \quad \Leftrightarrow \quad J_{ij} = \frac{\partial x_i}{\partial q_j}. \tag{35}$$

The vectors $(dq_1, 0, 0)$, $(0, dq_2, 0)$ and $(0, 0, dq_3)$ at point (q_1, q_2, q_3) in q-space are mapped to the vectors $(\partial\vec{\boldsymbol{r}}/\partial q_1)dq_1$, $(\partial\vec{\boldsymbol{r}}/\partial q_2)dq_2$, $(\partial\vec{\boldsymbol{r}}/\partial q_3)dq_3$. In component form this is

$$\begin{bmatrix} dx_1^{(1)} \\ dx_2^{(1)} \\ dx_3^{(1)} \end{bmatrix} = \begin{bmatrix} \partial x_1/\partial q_1 & \partial x_1/\partial q_2 & \partial x_1/\partial q_3 \\ \partial x_2/\partial q_1 & \partial x_2/\partial q_2 & \partial x_2/\partial q_3 \\ \partial x_3/\partial q_1 & \partial x_3/\partial q_2 & \partial x_3/\partial q_3 \end{bmatrix} \begin{bmatrix} dq_1 \\ 0 \\ 0 \end{bmatrix}, \tag{36}$$

where $(dx_1^{(1)}, dx_2^{(1)}, dx_3^{(1)})$ are the x-components of the vector $(\partial\vec{\boldsymbol{r}}/\partial q_1)dq_1$. Similar relations hold for the other basis vectors. Note that the rectangular box in q-space is in general mapped to a non-rectangular parallelepiped in x-space so the notation $dx_1 dx_2 dx_3$ for the volume element in (33) is a (common) abuse of notation.

The formulas (33), (34) tells us how to change variables in multiple integrals. This formula generalizes to higher dimension, and also to lower dimension. In the 2 variable case, we have a 2-by-2 determinant that can also be understood as a special case of the surface element formula (11) for a mapping $\vec{\boldsymbol{r}}(q_1, q_2)$ from a 2D space (q_1, q_2) to another 2D-space (x_1, x_2). In that case $\vec{\boldsymbol{r}}(q_1, q_2) = \vec{\boldsymbol{e}}_1 x_1(q_1, q_2) + \vec{\boldsymbol{e}}_2 x_2(q_1, q_2)$ so $(\partial\vec{\boldsymbol{r}}/\partial q_1) \times (\partial\vec{\boldsymbol{r}}/\partial q_2) dq_1 dq_2 = \vec{\boldsymbol{e}}_3\, dA_x$ and

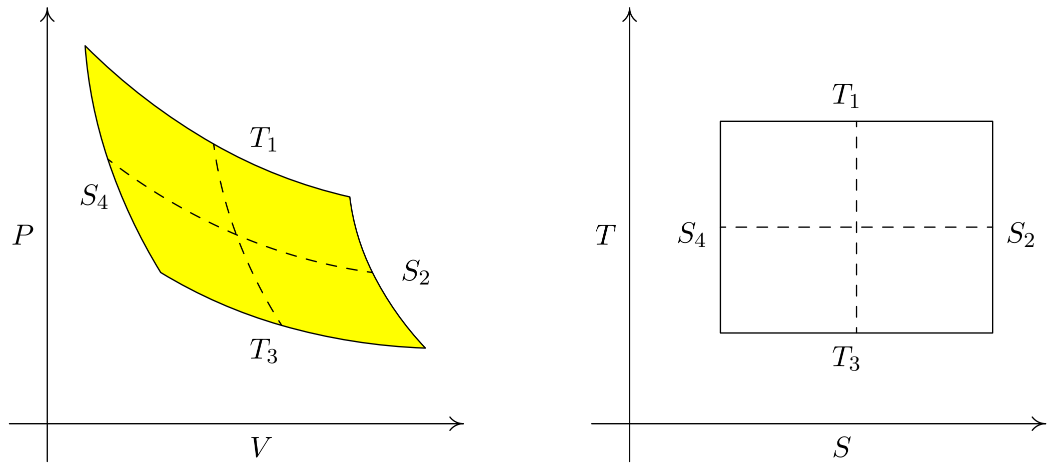
$$dx_1 dx_2 = dA_x = J\, dq_1 dq_2 = J\, dA_q \tag{37}$$

where the *Jacobian determinant* is now

$$J = \begin{vmatrix} \partial x_1/\partial q_1 & \partial x_1/\partial q_2 \\ \partial x_2/\partial q_1 & \partial x_2/\partial q_2 \end{vmatrix}. \tag{38}$$

1.7.1 Change of variables example

Consider a Carnot cycle for a perfect gas. The equation of state is $PV = nRT = NkT$, where P is pressure, V is the volume of gas and T is the temperature in Kelvins. The volume V contains n moles of gas corresponding to N molecules, R is the gas constant and k is Boltzmann's constant with $nR = Nk$. A *Carnot cycle* is an idealized thermodynamic cycle in which a gas goes through (1) a heated isothermal expansion at temperature T_1, (2) an adiabatic expansion at constant entropy S_2, (3) an isothermal compression releasing heat at temperature $T_3 < T_1$ and (4) an adiabatic compression at constant entropy $S_4 < S_2$. For a perfect monoatomic gas, constant entropy means constant PV^γ where $\gamma = C_P/C_V = 5/3$ with C_P and C_V the heat capacity at constant pressure P or constant volume V, respectively. Thus let $S = PV^\gamma$ (this S is not the physical entropy but it is constant whenever entropy is constant, we can call S a 'pseudo-entropy').



Now the work done by the gas when its volume changes from V_a to V_b is $\int_{V_a}^{V_b} P dV$ (since work = Force × displacement, P = force/area and V=area × displacement). Thus the (yellow) area inside the cycle in the (P, V) plane is the net work performed by the gas during one cycle. Although we can calculate that area by working in the (P, V) plane, it is easier to calculate it by using a change of variables from P, V to T, S. The area inside the cycle in the (P, V) plane is not the same as the area inside the cycle in the (S, T) plane. There is a distortion. An element of area in the (P, V) plane aligned with the T and S coordinates (*i.e.* with the dashed curves in the (P, V) plane) is

$$dA = \left| \left(\frac{\partial P}{\partial S} \vec{\boldsymbol{e}}_P + \frac{\partial V}{\partial S} \vec{\boldsymbol{e}}_V \right) dS \times \left(\frac{\partial P}{\partial T} \vec{\boldsymbol{e}}_P + \frac{\partial V}{\partial T} \vec{\boldsymbol{e}}_V \right) dT \right| \tag{39}$$

where $\vec{\boldsymbol{e}}_P$ and $\vec{\boldsymbol{e}}_V$ are the unit vectors in the P and V directions in the (P, V) plane, respectively. This is entirely similar to the area element of surface $\vec{\boldsymbol{r}}(u, v)$ being equal to $d\vec{\boldsymbol{S}} = \frac{\partial \vec{\boldsymbol{r}}}{\partial u} du \times \frac{\partial \vec{\boldsymbol{r}}}{\partial v} dv$ (but don't confuse the surface element $d\vec{\boldsymbol{S}}$ with the pseudo-entropy differential dS used in the present example!). Calculating out the cross product, we obtain

$$dA = \left| \left(\frac{\partial P}{\partial S} \frac{\partial V}{\partial T} - \frac{\partial V}{\partial S} \frac{\partial P}{\partial T} \right) dS dT \right| = \pm\, J_{S,T}^{P,V}\; dS dT \tag{40}$$

where the $\pm$ sign will be chosen to get a positive area (this depends on the bounds of integrations) and

$$J_{S,T}^{P,V} = \det(\boldsymbol{J}_{S,T}^{P,V}) = \begin{vmatrix} \dfrac{\partial P}{\partial S} & \dfrac{\partial P}{\partial T} \\[2ex] \dfrac{\partial V}{\partial S} & \dfrac{\partial V}{\partial T} \end{vmatrix} \tag{41}$$

is the *Jacobian* determinant (here the vertical bars are the common notation for determinants). It is the determinant of the Jacobian *matrix* $\boldsymbol{J}_{S,T}^{P,V}$ that corresponds to the mapping from (S, T) to

(P, V). The cycle area $A_{P,V}$ in the (P, V) plane is thus

$$A_{P,V} = \int_{T_3}^{T_1} \int_{S_4}^{S_2} \left| J_{S,T}^{P,V} \right| dS dT. \tag{42}$$

Note that the vertical bars in this formula are for absolute value of $J_{S,T}^{P,V}$ and the bounds have been selected so that $dS > 0$ and $dT > 0$ (in the limit-of-a-sum sense). This expression for the (P, V) area expressed in terms of (S, T) coordinates is simpler than if we used (P, V) coordinates, except for that $\left| J_{S,T}^{P,V} \right|$ since we do not have explicit expression for $P(S, T)$ and $V(S, T)$. What we have in fact are the inverse functions $T = PV/(Nk)$ and $S = PV^\gamma$. To find the partial derivatives that we need we could (1) find the inverse functions by solving for P and V in terms of T and S then compute the partial derivatives and the Jacobian, or (2) use implicit differentiation *e.g.* $Nk\, \partial T/\partial T = Nk = V \partial P/\partial T + P \partial V/\partial T$ and $\partial S/\partial T = 0 = V^\gamma \partial P/\partial T + P\gamma V^{\gamma-1} \partial V/\partial T$ etc. and solve for the partial derivatives we need. But there is a simpler way that makes use of an important property of Jacobians.

1.7.2 Geometric meaning of the Jacobian determinant and its inverse

The Jacobian $J_{S,T}^{P,V}$ represents the stretching factor of area elements when moving from the (S, T) plane to the (P, V) plane. If $dA_{S,T}$ is an area element centered at point (S, T) in the (S, T) plane then that area element gets mapped to an area element $dA_{P,V}$ centered at the corresponding point in the (P, V) plane. That's what equation (40) represents. In that equation we have in mind the mapping of a rectangular element of area $dSdT$ in the (S, T) plane to a parallelogram element in the (P, V) plane. The stretching factor is $|J_{S,T}^{P,V}|$ (as we saw earlier, the meaning of the sign is related to orientation, but here we are worrying only about areas, so we take absolute values). That relationship is valid for area elements of any shape, not just rectangles to parallelogram since the differential relationships implies an implicit limit-of-a-sum and in that limit, the 'pointwise' area stretching is independent of the shape of the area elements. A disk element in the (S, T) plane would be mapped to an ellipse element in the (P, V) plane but the pointwise area stretching would be the same as for a rectangular element (this is not true for *finite* size areas). So equation (40) can be written in the more general form

$$dA_{P,V} = |J_{S,T}^{P,V}|\, dA_{S,T} \tag{43}$$

which is locally valid for area elements of any shape. The key point is that if we consider the inverse map, back from (P, V) to (S, T) then there is an are stretching give by the Jacobian $J_{P,V}^{S,T} = (\partial S/\partial P)(\partial T/\partial V) - (\partial S/\partial V)(\partial T/\partial P)$ such that

$$dA_{S,T} = |J_{P,V}^{S,T}|\, dA_{P,V} \tag{44}$$

but since we are coming back to the original $dA_{S,T}$ element we must have

$$\boxed{J_{S,T}^{P,V} J_{P,V}^{S,T} = 1,} \tag{45}$$

so the Jacobian determinant are inverses of one another. This inverse relationship actually holds for the Jacobian *matrices* also

$$\boxed{\boldsymbol{J}_{S,T}^{P,V} \boldsymbol{J}_{P,V}^{S,T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.} \tag{46}$$

The latter can be derived from the chain rule since (note the consistent ordering of the partials)

$$\boldsymbol{J}_{S,T}^{P,V} = \begin{pmatrix} \partial P/\partial S & \partial P/\partial T \\ \partial V/\partial S & \partial V/\partial T \end{pmatrix}, \qquad \boldsymbol{J}_{P,V}^{S,T} = \begin{pmatrix} \partial S/\partial P & \partial S/\partial V \\ \partial T/\partial P & \partial T/\partial V \end{pmatrix} \tag{47}$$

and the matrix product of those two Jacobian matrices yields the identity matrix. For instance, the first row times the first column gives

$$\left(\frac{\partial P}{\partial S}\right)_T \left(\frac{\partial S}{\partial P}\right)_V + \left(\frac{\partial P}{\partial T}\right)_S \left(\frac{\partial T}{\partial P}\right)_V = \left(\frac{\partial P}{\partial P}\right)_V = 1.$$

A subscript has been added to remind which other variable is held fixed during the partial differentiation. The inverse relationship between the Jacobian determinants (45) then follows from the inverse relationship between the Jacobian matrices (46) since the determinant of a product is the product of the determinants. This important property of determinants can be verified directly by explicit calculation for this 2-by-2 case.
So what's the work done by the gas during one Carnot cycle? Well,

$$J_{S,T}^{P,V} = \left(J_{P,V}^{S,T}\right)^{-1} = \left(\frac{\partial S}{\partial P}\frac{\partial T}{\partial V} - \frac{\partial S}{\partial V}\frac{\partial T}{\partial P}\right)^{-1} = Nk\left(V^\gamma P - \gamma V^{\gamma-1} PV\right)^{-1} = \frac{Nk}{(1-\gamma)S} \tag{48}$$

so

$$A_{P,V} = \int_{T_3}^{T_1}\int_{S_4}^{S_2} \left|J_{S,T}^{P,V}\right| dSdT = \int_{T_3}^{T_1}\int_{S_4}^{S_2} \frac{Nk}{(\gamma-1)S} dSdT = \frac{Nk(T_1-T_3)}{(\gamma-1)} \ln\frac{S_2}{S_4}, \tag{49}$$

since $\gamma > 1$ and other quantities are positive.

Exercises

1. Calculate the area between the curves $xy = \alpha_1$, $xy = \alpha_2$ and $y = \beta_1 x$, $y = \beta_2 x$ in the (x, y) plane. Sketch the area. (α_1, α_2, β_1, $\beta_2 > 0$.)

2. Calculate the area between the curves $xy = \alpha_1$, $xy = \alpha_2$ and $y^2 = 2\beta_1 x$, $y^2 = 2\beta_2 x$ in the (x, y) plane. Sketch the area. (α_1, α_2, β_1, $\beta_2 > 0$.)

3. Calculate the area between the curves $x^2 + y^2 = 2\alpha_1 x$, $x^2 + y^2 = 2\alpha_2 x$ and $x^2 + y^2 = 2\beta_1 y$, $x^2 + y^2 = 2\beta_2 y$. Sketch the area. (α_1, α_2, β_1, $\beta_2 > 0$.)

4. Calculate the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by transforming them to a disk and a sphere, respectively, using a change of variables. [Hint: consider the change of variables $x = au$, $y = bv$, $z = cw$.]

5. Calculate the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$. Deduce the value of the Poisson integral $\int_{-\infty}^{\infty} e^{-x^2} dx$. [Hint: switch to polar coordinates].

6. Calculate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (a^2 + x^2 + y^2)^\alpha dxdy$. Where $a \neq 0$ and α is real. Discuss the values of α for which the integral exists.

2 Grad, div, curl

Consider a scalar function of a vector variable: $f(\vec{r})$, for instance the pressure $p(\vec{r})$ as a function of position, or the temperature $T(\vec{r})$ at point $\vec{r}$, etc. One way to visualize such functions is to consider *isosurfaces* or *level sets*, these are the set of all $\vec{r}$'s for which $f(\vec{r}) = C_0$, for some constant C_0. In cartesian coordinates $\vec{r} = x\,\vec{e} + y\,\hat{y} + z\,\hat{z}$ and the scalar function is a function of the three coordinates $f(\vec{r}) \equiv f(x, y, z)$, hence we can interpret an isosurface $f(x, y, z) = C_0$ as a single equation for the 3 unknowns x, y, z. In general, we are free to pick two of those variables, x and y say, then solve the equation $f(x, y, z) = C_0$ for z. For example, the isosurfaces of $f(\vec{r}) = x^2 + y^2 + z^2$ are determined by the equation $f(\vec{r}) = C_0$. This is the sphere or radius $\sqrt{C_0}$, if $C_0 \geq 0$.

2.1 Geometric definition of the Gradient

The value of $f(\vec{r})$ at a point $\vec{r}_0$ defines an isosurface $f(\vec{r}) = f(\vec{r}_0)$ through that point $\vec{r}_0$. The *gradient* of $f(\vec{r})$ at $\vec{r}_0$ can be defined geometrically as the *vector*, denoted $\vec{\nabla} f(\vec{r}_0)$, that

(i) is perpendicular to the isosurface $f(\vec{r}) = f(\vec{r}_0)$ at the point $\vec{r}_0$ and points in the direction of increase of $f(\vec{r})$ and

(ii) has a magnitude equal to the rate of change of $f(\vec{r})$ with distance from the isosurface.

From this geometric definition, we deduce that the plane tangent to the isosurface $f(\vec{r}) = f(\vec{r}_0)$ at $\vec{r}_0$ has the equation

$$(\vec{r} - \vec{r}_0) \cdot \vec{\nabla} f(\vec{r}_0) = 0 \tag{50}$$

Likewise, the plane $(\vec{r} - \vec{r}_0) \cdot \vec{\nabla} f(\vec{r}_0) = \epsilon$ is parallel to the tangent plane but further 'up' in the direction of the gradient if $\epsilon > 0$ or 'down' if $\epsilon < 0$. More generally, the function $f(\vec{r})$ can be locally (*i.e.* for $\vec{r}$ near $\vec{r}_0$) approximated by

$$f(\vec{r}) \approx f(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \vec{\nabla} f(\vec{r}_0). \tag{51}$$

Indeed since $\vec{\nabla} f$ points in the direction of fastest increase of $f(\vec{r})$ and has a magnitude equal to the rate of change of f with distance in that direction, the change in f as we move slightly away from $\vec{r}_0$ only depends on how much we have moved in the direction of the gradient. The actual change in f is the distance times the rate of change with distance, hence it is $(\vec{r} - \vec{r}_0) \cdot \vec{\nabla} f(\vec{r}_0)$. Equation (51) states that the isosurfaces of $f(\vec{r})$ look like planes perpendicular to $\vec{\nabla} f(\vec{r}_0)$ for $\vec{r}$ in the neighborhood for $\vec{r}_0$. Equation (51) is a *linear approximation* of $f(\vec{r})$ in the neighborhood of $\vec{r}_0$, just like $f(x) \approx f(x_0) + (x - x_0)f'(x_0)$ for functions of one variable. This is not exact, there is a small error that goes to zero faster than $|\vec{r} - \vec{r}_0|$ as $\vec{r} \to \vec{r}_0$. The limit as $\vec{r} \to \vec{r}_0$ can be written as the exact differential relation

$$\boxed{df(\vec{r}) = d\vec{r} \cdot \vec{\nabla} f(\vec{r}),} \tag{52}$$

where $df(\vec{r}) = f(\vec{r} + d\vec{r}) - f(\vec{r})$ is the differential change in $f(\vec{r})$ when $\vec{r}$ changes from $\vec{r}$ to $\vec{r} + d\vec{r}$. This differential relationship holds for any $\vec{r}$ and $d\vec{r}$. It is analogous to the differential relationship $df = f'(x)\,dx$ for functions of a single variable.

► It follows immediately from the geometric definition of the gradient that if $r = |\vec{r}|$ is the distance to the origin and $\hat{r} = \vec{r}/r$ is the unit radial vector, then $\vec{\nabla} r = \hat{r}$ and more generally $\vec{\nabla} f(r) = \hat{r}\,df/dr$. For instance, $\vec{\nabla}(1/r) = -\hat{r}/r^2$. ■

2.2 Directional derivative, gradient and the $\vec{\boldsymbol{\nabla}}$ operator

The rate of change of $f(\vec{\boldsymbol{r}})$ in the direction of the unit vector $\hat{\boldsymbol{n}}$ at point $\vec{\boldsymbol{r}}$, denoted $\partial f(\vec{\boldsymbol{r}})/\partial n$ is defined as

$$\frac{\partial f(\vec{\boldsymbol{r}})}{\partial n}=\lim_{h\to 0}\frac{f(\vec{\boldsymbol{r}}+h\hat{\boldsymbol{n}})-f(\vec{\boldsymbol{r}})}{h}. \tag{53}$$

From (51) and (52),

$$\frac{\partial f(\vec{\boldsymbol{r}})}{\partial n}=\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\nabla}}f(\vec{\boldsymbol{r}}). \tag{54}$$

This is exact because the error of the linear approximation (51), $f(\vec{\boldsymbol{r}}+h\hat{\boldsymbol{n}})\approx f(\vec{\boldsymbol{r}})+h\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\nabla}}f(\vec{\boldsymbol{r}})$, goes to zero faster than h as $h\to 0$.
In particular,

$$\frac{\partial f}{\partial x}=\hat{\boldsymbol{x}}\cdot\vec{\boldsymbol{\nabla}}f,\quad \frac{\partial f}{\partial y}=\hat{\boldsymbol{y}}\cdot\vec{\boldsymbol{\nabla}}f,\quad \frac{\partial f}{\partial z}=\hat{\boldsymbol{z}}\cdot\vec{\boldsymbol{\nabla}}f, \tag{55}$$

hence

$$\vec{\boldsymbol{\nabla}}f=\hat{\boldsymbol{x}}\frac{\partial f}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial f}{\partial y}+\hat{\boldsymbol{z}}\frac{\partial f}{\partial z}, \tag{56}$$

for cartesian coordinates x, y, z. We can also deduce this important result from the *Chain rule* for functions of several variables. In cartesian coordinates, written (x_1,x_2,x_3) in place of (x,y,z), the position vector reads $\vec{\boldsymbol{r}}=x_1\vec{\boldsymbol{e}}_1+x_2\vec{\boldsymbol{e}}_2+x_3\vec{\boldsymbol{e}}_3$ and the scalar function $f(\vec{\boldsymbol{r}})\equiv f(x_1,x_2,x_3)$ is a function of the 3 coordinates. The directional derivative

$$\frac{\partial f(\vec{\boldsymbol{r}})}{\partial n}=\lim_{h\to 0}\frac{f(\vec{\boldsymbol{r}}+h\hat{\boldsymbol{n}})-f(\vec{\boldsymbol{r}})}{h}=n_1\frac{\partial f}{\partial x_1}+n_2\frac{\partial f}{\partial x_2}+n_3\frac{\partial f}{\partial x_3}=\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\nabla}}f. \tag{57}$$

This follows from rewriting the difference $f(\vec{\boldsymbol{r}}+h\hat{\boldsymbol{n}})-f(\vec{\boldsymbol{r}})$ as the telescoping sum

$$\begin{aligned}&[f(x_1+hn_1,x_2+hn_2,x_3+hn_1)-f(x_1,x_2+hn_2,x_3+hn_3)]\\&\quad+[f(x_1,x_2+hn_2,x_3+hn_3)-f(x_1,x_2,x_3+hn_3)]\\&\qquad+[f(x_1,x_2,x_3+hn_3)-f(x_1,x_2,x_3)]\end{aligned} \tag{58}$$

then using continuity and the definition of the partial derivatives, *e.g.*

$$\lim_{h\to 0}\frac{f(\vec{\boldsymbol{r}}+hn_3\vec{\boldsymbol{e}}_3)-f(\vec{\boldsymbol{r}})}{h}=n_3\lim_{\delta\to 0}\frac{f(\vec{\boldsymbol{r}}+\delta\vec{\boldsymbol{e}}_3)-f(\vec{\boldsymbol{r}})}{\delta}=n_3\frac{\partial f}{\partial x_3}.$$

Either way, we establish that in cartesian coordinates

$$\boxed{\vec{\boldsymbol{\nabla}}=\vec{\boldsymbol{e}}_1\frac{\partial}{\partial x_1}+\vec{\boldsymbol{e}}_2\frac{\partial}{\partial x_2}+\vec{\boldsymbol{e}}_3\frac{\partial}{\partial x_3}\equiv\vec{\boldsymbol{e}}_i\partial_i} \tag{59}$$

where ∂_i is short for $\partial/\partial x_i$ and we have used the convention of summation over all values of the repeated index i.

2.3 Div and Curl

We'll depart from our geometric point of view to first define divergence and curl computationally based on their cartesian representation. Here we consider *vector fields* $\vec{\boldsymbol{v}}(\vec{\boldsymbol{r}})$ which are vector functions of a vector variable, for example the velocity $\vec{\boldsymbol{v}}(\vec{\boldsymbol{r}})$ of a fluid at point $\vec{\boldsymbol{r}}$, or the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}})$ at point $\vec{\boldsymbol{r}}$, etc. For the geometric meaning of divergence and curl, see the sections on divergence and Stokes' theorems.

The *divergence* of a vector field $\vec{v}(\vec{r})$ is defined as the dot product $\vec{\nabla}\cdot\vec{v}$. Now since the unit vectors $\vec{e}_i$ are constant for cartesian coordinates, $\vec{\nabla}\cdot\vec{v} = (\vec{e}_i\partial_i)\cdot(\vec{e}_j v_j) = (\vec{e}_i\cdot\vec{e}_j)\,\partial_i v_j = \delta_{ij}\partial_i v_j$ hence

$$\boxed{\vec{\nabla}\cdot\vec{v} = \partial_i v_i = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3.} \tag{60}$$

Likewise, the *curl* of a vector field $\vec{v}(\vec{r})$ is the cross product $\vec{\nabla}\times\vec{v}$. In cartesian coordinates, $\vec{\nabla}\times\vec{v} = (\vec{e}_j\partial_j)\times(\vec{e}_k v_k) = (\vec{e}_j\times\vec{e}_k)\,\partial_j v_k$. Recall that $\epsilon_{ijk} = \vec{e}_i\cdot(\vec{e}_j\times\vec{e}_k)$, or in other words ϵ_{ijk} is the i component of the vector $\vec{e}_j\times\vec{e}_k$, thus $\vec{e}_j\times\vec{e}_k = \vec{e}_i\,\epsilon_{ijk}$ and

$$\boxed{\vec{\nabla}\times\vec{v} = \vec{e}_i\,\epsilon_{ijk}\,\partial_j v_k.} \tag{61}$$

(Recall that $\vec{a}\times\vec{b} = \vec{e}_i\epsilon_{ijk}a_j b_k$.) The right hand side is a *triple sum* over all values of the repeated indices i, j and k! But that triple sum is not too bad since $\epsilon_{ijk} = \pm 1$ depending on whether (i,j,k) is a cyclic (=even) permutation or an acyclic (=odd) permutation of $(1,2,3)$ and vanishes in all other instances. Thus (61) expands to

$$\vec{\nabla}\times\vec{v} = \vec{e}_1\,(\partial_2 v_3 - \partial_3 v_2) + \vec{e}_2\,(\partial_3 v_1 - \partial_1 v_3) + \vec{e}_3\,(\partial_1 v_2 - \partial_2 v_1)\,. \tag{62}$$

We can also write that the i-th cartesian component of the curl is

$$\boxed{\vec{e}_i\cdot\left(\vec{\nabla}\times\vec{v}\right) = \left(\vec{\nabla}\times\vec{v}\right)_i = \epsilon_{ijk}\,\partial_j v_k.} \tag{63}$$

Note that the divergence is a scalar but the curl is a vector.

2.4 Vector identities

In the following $f = f(\vec{r})$ is an arbitrary scalar field while $\vec{v}(\vec{r})$ and $\vec{w}(\vec{r})$ are vector fields. Two fundamental identities that can be remembered from $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ and $\vec{a}\times(\alpha\vec{a}) = 0$ are

$$\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0,} \tag{64}$$

and

$$\boxed{\vec{\nabla}\times(\vec{\nabla}f) = 0.} \tag{65}$$

The divergence of a curl and the curl of a gradient vanish identically (assuming all those derivatives exist). These can be proved using index notation. Other useful identities are

$$\vec{\nabla}\cdot(f\vec{v}) = (\vec{\nabla}f)\cdot\vec{v} + f(\vec{\nabla}\cdot\vec{v}), \tag{66}$$

$$\vec{\nabla}\times(f\vec{v}) = (\vec{\nabla}f)\times\vec{v} + f(\vec{\nabla}\times\vec{v}), \tag{67}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \nabla^2\vec{v}, \tag{68}$$

where $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the *Laplacian* operator.
The identity (66) is verified easily using indicial notation

$$\vec{\nabla}\cdot(f\vec{v}) = \partial_i(f v_i) = (\partial_i f)v_i + f(\partial_i v_i) = (\vec{\nabla}f)\cdot\vec{v} + f(\vec{\nabla}\cdot\vec{v}).$$

Likewise the second identity (67) follows from $\epsilon_{ijk}\partial_j(f v_k) = \epsilon_{ijk}(\partial_j f)v_k + f\epsilon_{ijk}(\partial_j v_k)$. The identity (68) is easily remembered from the double cross product formula $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ but note that the $\vec{\nabla}$ in the first term *must* appear first since $\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) \neq (\vec{\nabla}\cdot\vec{v})\vec{\nabla}$. That identity can be verified using indicial notation if one knows the double cross product identity in terms of the permutation tensor (see earlier notes on ϵ_{ijk} and index notation) namely

$$\boxed{\epsilon_{ijk}\epsilon_{klm} = \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.} \tag{69}$$

A slightly trickier identity is

$$\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}) = (\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}} - (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}})\vec{\boldsymbol{w}} + (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{w}})\vec{\boldsymbol{v}} - (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{w}}, \tag{70}$$

where $(\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{v}} = (w_j\partial_j)v_i$ in indicial notation. This can be verified using (69) and can be remembered from the double cross-product identity with the additional input that $\vec{\boldsymbol{\nabla}}$ is a vector *operator*, not just a regular vector, hence $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}})$ represents derivatives of a product and this doubles the number of terms of the resulting expression. The first two terms are the double cross product $\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}})\vec{\boldsymbol{b}} - (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}})\vec{\boldsymbol{c}}$ for derivatives of $\vec{\boldsymbol{v}}$ while the last two terms are the double cross product for derivatives of $\vec{\boldsymbol{w}}$.
Another similar identity is

$$\vec{\boldsymbol{v}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{w}}) = (\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{w}}) \cdot \vec{\boldsymbol{v}} - (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{w}}. \tag{71}$$

In indicial notation this reads

$$\epsilon_{ijk}v_j\,(\epsilon_{klm}\partial_l w_m) = (\partial_i w_j)v_j - (v_j\partial_j)w_i. \tag{72}$$

Note that this last identity involves the gradient of a vector field $\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{w}}$. This makes sense and is a *tensor*, *i.e.* a geometric object whose components with respect to a basis form a *matrix*. In indicial notation, the components of $\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{w}}$ are $\partial_i w_j$ and there are 9 of them. This is very different from $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{w}} = \partial_i w_i$ which is a scalar.
The bottom line is that these identities can be reconstructed relatively easily from our knowledge of regular vector identities for dot, cross and double cross products, however $\vec{\boldsymbol{\nabla}}$ is a vector of derivatives and one needs to watch out more carefully for the order and the product rules. If in doubt, jump to indicial notation.

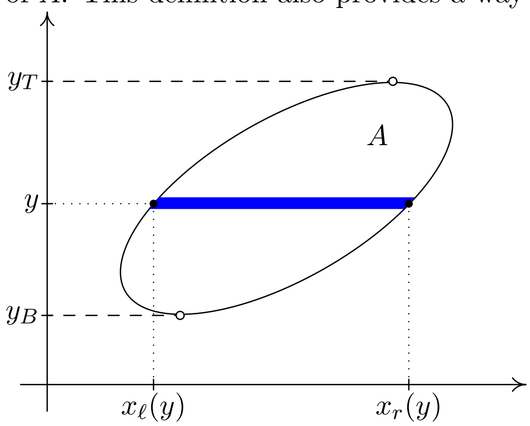
Exercises

1. Verify (64) and (65).
2. Digest and verify the identity (69) *ab initio.*
3. Verify the double cross product identities $\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \vec{\boldsymbol{b}}(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}) - \vec{\boldsymbol{c}}(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}})$ and $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \times \vec{\boldsymbol{c}} = \vec{\boldsymbol{b}}(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}) - \vec{\boldsymbol{a}}(\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{c}})$ in indicial notation using (69).
4. Verify (70) and (71) using indicial notation and (69).
5. Use (69) to derive vector identities for $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) \cdot (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{d}})$ and $(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{w}})$.
6. Show that $\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}) = \vec{\boldsymbol{w}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) - \vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{w}})$ and explain how to reconstruct this from the rules for the mixed (or box) product $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}})$ of three regular vectors.
7. Find the fastest way to show that $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{r}} = 3$ and $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{r}} = 0$.
8. Find the fastest way to show that $\vec{\boldsymbol{\nabla}} \cdot (\hat{\boldsymbol{r}}/r^2) = 0$ and $\vec{\boldsymbol{\nabla}} \times (\hat{\boldsymbol{r}}/r^2) = 0$ for all $\vec{\boldsymbol{r}} \neq 0$.
9. Quickly calculate $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}$ when $\vec{\boldsymbol{B}} = (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}})/|\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}}|^2$ (*cf.* Biot-Savart law for a line current) [Hint: use both vector identities and cartesian coordinates where convenient].

3 Fundamental theorems of vector calculus

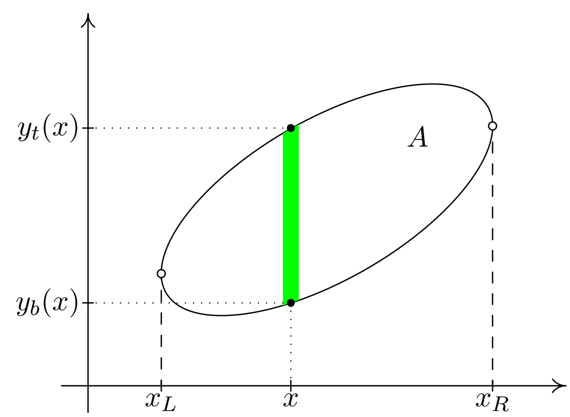
3.1 Integration in $\mathbb{R}^2$ and $\mathbb{R}^3$

The integral of a function of two variables $f(x,y)$ over a domain A of $\mathbb{R}^2$ denoted $\int_A f(x,y)dA$ can be defined as a the limit of a $\sum_n f(x_n,y_n)\Delta A_n$ where the A_n's, $n=1,\ldots,N$ provide an (approximate) partition of A that breaks up A into a set of small area elements, squares or triangles for instance. ΔA_n is the area of those element n and (x_n,y_n) is a point inside that element, for instance the center of area of the triangle. The integral would be the limit of such sums when the area of the triangles goes to zero and their number N must then go to infinity. This limit should be such that the aspect ratios of the triangles remain bounded away from 0 so we get a finer and finer sampling of A. This definition also provides a way to approximate the integral by such a finite sum.



If we imagine breaking up A into small squares aligned with the x and y axes then the sum over all squares inside A can be performed row by row. Each row-sum then tends to an integral in the x-direction, this leads to the conclusion that the integral can be calculated as *iterated* integrals

$$\int_A f(x,y)dA = \int_{y_B}^{y_T} dy \int_{x_\ell(y)}^{x_r(y)} f(x,y)dx \tag{73}$$



We can also imagine summing up column by column instead and each column-sum then tends to an integral in the y-direction, this leads to the *iterated* integrals

$$\int_A f(x,y)dA = \int_{x_L}^{x_R} dx \int_{y_b(x)}^{y_t(x)} f(x,y)dy. \tag{74}$$

Note of course that the limits of integrations differ from those of the previous iterated integrals.

This iterated integral approach readily extends to integrals over three-dimensional domains in $\mathbb{R}^3$ and more generally to integrals in $\mathbb{R}^n$.

3.2 Fundamental theorem of Calculus

The fundamental theorem of calculus can be written

$$\boxed{\int_a^b \frac{dF}{dx}dx = F(b)-F(a).} \tag{75}$$

Once again we can interpret this in terms of limits of finite differences. The derivative is defined as

$$\frac{dF}{dx} = \lim_{\Delta x\to 0}\frac{\Delta F}{\Delta x} \tag{76}$$

where $\Delta F = F(x+\Delta x) - F(x)$, while the integral

$$\int_a^b f(x)dx = \lim_{\Delta x_n \to 0} \sum_{n=1}^{N} f(\tilde{x}_n)\Delta x_n \tag{77}$$

where $\Delta x_n = x_n - x_{n-1}$ and $x_{n-1} \leq \tilde{x}_n \leq x_n$, with $n = 1,\ldots,N$ and $x_0 = a$, $x_N = b$, so the set of x_n's provides a partition of the interval $[a,b]$. The best choice for $\tilde{x}_n$ is the midpoint $\tilde{x}_n = (x_n + x_{n-1})/2$. This is the *midpoint* scheme in numerical integration methods. Putting these two limits together and setting $\Delta F_n = F(x_n) - F(x_{n-1})$ we can write

$$\int_a^b \frac{dF}{dx}dx = \lim_{\Delta x_n \to 0} \sum_{n=1}^{N} \frac{\Delta F_n}{\Delta x_n}\Delta x_n = \lim_{\Delta x_n \to 0} \sum_{n=1}^{N} \Delta F_n = F(b) - F(a). \tag{78}$$

We can also write this in the integral form

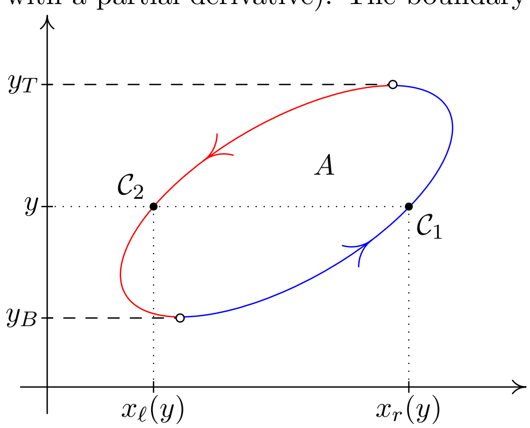
$$\int_a^b \frac{dF}{dx}dx = \int_{F(a)}^{F(b)} dF = F(b) - F(a). \tag{79}$$

3.3 Fundamental theorem in $\mathbb{R}^2$

From the fundamental theorem of calculus and the reduction of integrals on a domain A of $\mathbb{R}^2$ to iterated integrals on intervals in $\mathbb{R}$ we obtain for a function $G(x,y)$

$$\int_A \frac{\partial G}{\partial x}dA = \int_{y_B}^{y_T} dy \int_{x_\ell(y)}^{x_r(y)} \frac{\partial G}{\partial x}dx = \int_{y_B}^{y_T} \big[G(x_r(y),y) - G(x_\ell(y),y)\big]dy. \tag{80}$$

This looks nice enough but we can rewrite the integral on the right-hand side as a line integral over the boundary of A. The boundary of A is a closed curve $\mathcal{C}$ often denoted ∂A (not to be confused with a partial derivative). The boundary $\mathcal{C}$ has two parts $\mathcal{C}_1$ and $\mathcal{C}_2$.



The curve $\mathcal{C}_1$ can be parametrized in terms of y as $\vec{\boldsymbol{r}}(y) = \hat{\boldsymbol{x}}x_r(y) + \hat{\boldsymbol{y}}y$ with $y = y_B \to y_T$, hence

$$\int_{\mathcal{C}_1} G(x,y)\hat{\boldsymbol{y}}\cdot d\vec{\boldsymbol{r}} = \int_{y_B}^{y_T} G(x_r(y),y)dy.$$

Likewise, the curve $\mathcal{C}_2$ can be parametrized using y as $\vec{\boldsymbol{r}}(y) = \hat{\boldsymbol{x}}x_\ell(y) + \hat{\boldsymbol{y}}y$ with $y = y_T \to y_B$, hence

$$\int_{\mathcal{C}_2} G(x,y)\hat{\boldsymbol{y}}\cdot d\vec{\boldsymbol{r}} = \int_{y_T}^{y_B} G(x_\ell(y),y)dy.$$

Putting these two results together the right hand side of (80) becomes

$$\int_{y_B}^{y_T} \big[G(x_r(y),y) - G(x_\ell(y),y)\big]dy = \int_{\mathcal{C}_1} G\,\hat{\boldsymbol{y}}\cdot d\vec{\boldsymbol{r}} + \int_{\mathcal{C}_2} G\,\hat{\boldsymbol{y}}\cdot d\vec{\boldsymbol{r}} = \oint_{\mathcal{C}} G\,\hat{\boldsymbol{y}}\cdot d\vec{\boldsymbol{r}},$$

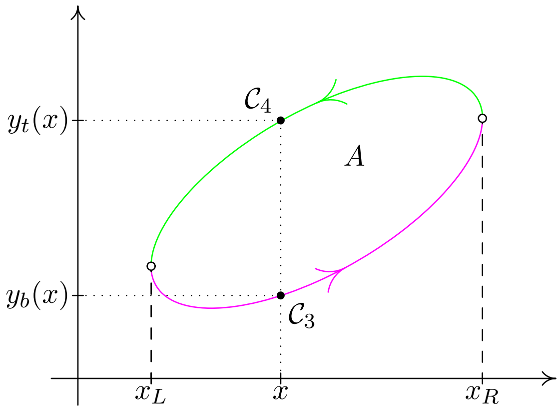
where $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ is the *closed* curve bounding A. Then (80) becomes

$$\int_A \frac{\partial G}{\partial x}dA = \oint_{\mathcal{C}} G\,\hat{\boldsymbol{y}}\cdot d\vec{\boldsymbol{r}}. \tag{81}$$

The symbol $\oint$ is used to emphasize that the integral is over a closed curve. Note that the curve $\mathcal{C}$ has been oriented *counter-clockwise* such that the interior is to the left of the curve.
Similarly the fundamental theorem of calculus and iterated integrals lead to the result that

$$\int_A \frac{\partial F}{\partial y} dA = \int_{x_L}^{x_R} dx \int_{y_b(x)}^{y_t(x)} \frac{\partial F}{\partial y} dy = \int_{x_L}^{x_R} \left[F(x, y_t(x)) - F(x, y_b(x))\right] dx, \tag{82}$$

and the integral on the right hand side can be rewritten as a line integral around the boundary curve $\mathcal{C} = \mathcal{C}_3 + \mathcal{C}_4$.



The curve $\mathcal{C}_3$ can be parametrized in terms of x as $\vec{r}(x) = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}\, y_b(x)$ with $x = x_L \to x_R$, hence

$$\int_{\mathcal{C}_3} F(x, y)\hat{\boldsymbol{x}} \cdot d\vec{r} = \int_{x_L}^{x_R} F(x, y_b(x))dx.$$

Likewise, the curve $\mathcal{C}_4$ can be parametrized using x as $\vec{r}(x) = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}\, y_t(x)$ with $x = x_R \to x_L$, hence

$$\int_{\mathcal{C}_4} F(x, y)\hat{\boldsymbol{x}} \cdot d\vec{r} = \int_{x_R}^{x_L} F(x, y_t(x))dx.$$

The right hand side of (82) becomes

$$\int_{x_L}^{x_R} \left[F(x, y_t(x)) - F(x, y_b(x))\right] dx = -\int_{\mathcal{C}_4} F\hat{\boldsymbol{x}} \cdot d\vec{r} - \int_{\mathcal{C}_3} F\hat{\boldsymbol{x}} \cdot d\vec{r} = -\oint_{\mathcal{C}} F\hat{\boldsymbol{x}} \cdot d\vec{r},$$

where $\mathcal{C} = \mathcal{C}_3 + \mathcal{C}_4$ is the *closed* curve bounding A oriented *counter-clockwise* as before. Then (82) becomes

$$\int_A \frac{\partial F}{\partial y} dA = -\oint_{\mathcal{C}} F\hat{\boldsymbol{x}} \cdot d\vec{r}. \tag{83}$$

3.4 Green and Stokes' theorems

The two results (81) and (83) can be combined into a single important formula. Subtract (83) from (81) to deduce the *curl form of Green's theorem*

$$\boxed{\int_A \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dA = \oint_{\mathcal{C}} (F\hat{\boldsymbol{x}} + G\hat{\boldsymbol{y}}) \cdot d\vec{r} = \oint_{\mathcal{C}} (F dx + G dy).} \tag{84}$$

Note that dx and dy in the last integral are *not* independent quantities, they are the projection of the line element $d\vec{r}$ onto the basis vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ as written in the middle line integral. If $(x(t), y(t))$ is a parametrization for the curve then $dx = (dx/dt)dt$ and $dy = (dy/dt)dt$ and the t-bounds of integration should be picked to correspond to counter-clockwise orientation. Note also that Green's theorem (84) is the formula to remember since it includes both (81) when $F = 0$ and (83) when $G = 0$.
Green's theorem can be written in several equivalent forms. Define the vector field $\vec{v} = F(x, y)\hat{\boldsymbol{x}} + G(x, y)\hat{\boldsymbol{y}}$. A simple calculation verifies that its curl is purely in the $\hat{\boldsymbol{z}}$ direction indeed (62) gives $\vec{\nabla} \times \vec{v} = \hat{\boldsymbol{z}}\,(\partial G/\partial x - \partial F/\partial y)$ thus (84) can be rewritten in the form

$$\int_A \left(\vec{\nabla} \times \vec{v}\right) \cdot \hat{\boldsymbol{z}}\, dA = \oint_{\mathcal{C}} \vec{v} \cdot d\vec{r}, \tag{85}$$

This result also applies to any 3D vector field $\vec{\boldsymbol{v}}(x,y,z) = F(x,y,z)\hat{\boldsymbol{x}} + G(x,y,z)\hat{\boldsymbol{y}} + H(x,y,z)\hat{\boldsymbol{z}}$ and any planar surface A perpendicular to $\hat{\boldsymbol{z}}$ since $\hat{\boldsymbol{z}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}})$ still equals $\partial G/\partial x - \partial F/\partial y$ for such 3D vector fields and the line element $d\vec{\boldsymbol{r}}$ of the boundary curve $\mathcal{C}$ of such planar area is perpendicular to $\hat{\boldsymbol{z}}$ so $\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}}$ is still equal to $F(x,y,z)dx + G(x,y,z)dy$. The extra z coordinate is a mere parameter for the integrals and (85) applies equally well to 3D vector field $\vec{\boldsymbol{v}}(x,y,z)$ provided A is a planar area perpendicular to $\hat{\boldsymbol{z}}$.

In fact that last restriction on A itself can be removed. This is *Stokes' theorem* which reads

$$\boxed{\int_S \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}\right) \cdot d\vec{\boldsymbol{S}} = \oint_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}},} \tag{86}$$

where S is a bounded orientable surface in 3D space, not necessarily planar, and $\mathcal{C}$ is its closed curve boundary. The orientation of the surface as determined by the direction of its normal $\hat{\boldsymbol{n}}$, where $d\vec{\boldsymbol{S}} = \hat{\boldsymbol{n}} dS$, and the orientation of the boundary curve $\mathcal{C}$ must obey the right-hand rule. Thus a corkscrew turning in the direction of $\mathcal{C}$ would go through S in the direction of its normal $\hat{\boldsymbol{n}}$. That restriction is a direct consequence of the fact that the right hand rule enters the definition of the curl as the cross product $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$.

Stokes' theorem (86) provides a geometric interpretation for the curl. At any point $\vec{\boldsymbol{r}}$, consider a small disk of area A perpendicular to an arbitrary unit vector $\hat{\boldsymbol{n}}$, then Stokes' theorem states that

$$(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot \hat{\boldsymbol{n}} = \lim_{A \to 0} \frac{1}{A} \oint_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}} \tag{87}$$

where $\mathcal{C}$ is the circle bounding the disk A oriented with $\hat{\boldsymbol{n}}$. The line integral $\oint \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}}$ is called the *circulation* of the vector field $\vec{\boldsymbol{v}}$ around the closed curve $\mathcal{C}$.

Partial proof of Stokes' Theorem

► We'll sketch a proof of Stokes' theorem and do a non-trivial exercise in vector calculus in the process. Imagine a bounded surface S in 3D space that can be parametrized in terms of cartesian coordinates (x,y), that is $S: z = h(x,y)$ with x,y in some domain A in the (x,y)-plane then $\vec{\boldsymbol{r}}(x,y) = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}h(x,y)$ and

$$\frac{\partial \vec{\boldsymbol{r}}}{\partial x} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{z}}\frac{\partial h}{\partial x}, \quad \frac{\partial \vec{\boldsymbol{r}}}{\partial y} = \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\frac{\partial h}{\partial y}, \tag{88}$$

so

$$d\vec{\boldsymbol{S}} = \left(\frac{\partial \vec{\boldsymbol{r}}}{\partial x} \times \frac{\partial \vec{\boldsymbol{r}}}{\partial y}\right) dxdy = \left(\hat{\boldsymbol{z}} - \hat{\boldsymbol{x}}\frac{\partial h}{\partial x} - \hat{\boldsymbol{y}}\frac{\partial h}{\partial y}\right) dxdy. \tag{89}$$

Now if we write $\vec{\boldsymbol{v}} = \hat{\boldsymbol{x}}u + \hat{\boldsymbol{y}}v + \hat{\boldsymbol{z}}w$ then from (62)

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}} = \hat{\boldsymbol{x}}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{\boldsymbol{y}}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{\boldsymbol{z}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

and from (89)

$$\begin{aligned}(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot \left(\frac{\partial \vec{\boldsymbol{r}}}{\partial x} \times \frac{\partial \vec{\boldsymbol{r}}}{\partial y}\right) &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \frac{\partial h}{\partial y}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ &= \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial h}{\partial x}\right) - \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial h}{\partial y}\right) + \left(\frac{\partial h}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial w}{\partial y}\right).\end{aligned} \tag{90}$$

Now each of these expressions are partial derivatives of functions of three variables, thus $\partial v/\partial x$ for instance means derivative with respect to x with y and z fixed. However in the surface integral (86), all these expressions must be evaluated on the surface S, meaning at $z = h(x, y)$ with x, y in A. The *chain rule* then says that

$$\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial h}{\partial x}\right)_S = \frac{\partial \bar{v}}{\partial x} \quad \text{and} \quad \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial h}{\partial y}\right)_S = \frac{\partial \bar{u}}{\partial y} \tag{91}$$

where

$$\bar{u}(x, y) = u(x, y, h(x, y)), \quad \bar{v}(x, y) = v(x, y, h(x, y)). \tag{92}$$

and the subscript S indicates that the expression must be evaluated on S meaning at $z = h(x, y)$ with x, y in A. We can massage the last term in parentheses on the right hand side of (90)

$$\begin{aligned}\frac{\partial h}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial w}{\partial y} =& \frac{\partial h}{\partial y}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial h}{\partial x}\right) - \frac{\partial h}{\partial x}\left(\frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial h}{\partial y}\right)\\ =& \frac{\partial h}{\partial y}\frac{\partial \bar{w}}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial \bar{w}}{\partial y}\\ =& \frac{\partial}{\partial x}\left(\bar{w}\frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial y}\left(\bar{w}\frac{\partial h}{\partial x}\right).\end{aligned} \tag{93}$$

where $\bar{w}(x, y) = w(x, y, h(x, y))$ as in (92) and the last step follows from the cancellation of the mixed partials $\partial^2 h/\partial y \partial x = \partial^2 h/\partial x \partial y$. Substituting (91) and (93) into (90) yields

$$\begin{aligned}(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot \left(\frac{\partial \vec{\boldsymbol{r}}}{\partial x} \times \frac{\partial \vec{\boldsymbol{r}}}{\partial y}\right) =& \frac{\partial}{\partial x}\left(\bar{v} + \bar{w}\frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial y}\left(\bar{u} + \bar{w}\frac{\partial h}{\partial x}\right)\\ \equiv& \quad \frac{\partial G(x, y)}{\partial x} \quad - \quad \frac{\partial F(x, y)}{\partial y}.\end{aligned} \tag{94}$$

Hurrah! we have managed to write

$$\int_S \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}\right) \cdot d\vec{\boldsymbol{S}} = \int_A \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dA. \tag{95}$$

Now for any point $\vec{\boldsymbol{r}}(x, y)$ on the surface S, we have from (88)

$$d\vec{\boldsymbol{r}} = \frac{\partial \vec{\boldsymbol{r}}}{\partial x}dx + \frac{\partial \vec{\boldsymbol{r}}}{\partial y}dy = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy + \hat{\boldsymbol{z}}\left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)$$

where dx and dy would be independent for any point on S except on its boundary $\mathcal{C}$ where dx and dy must be such that $d\vec{\boldsymbol{r}}$ is tangent to $\mathcal{C}$. In any case, at any point on the surface S

$$\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}} = \left(\bar{u} + \bar{w}\frac{\partial h}{\partial x}\right) dx + \left(\bar{v} + \bar{w}\frac{\partial h}{\partial y}\right) dy = F(x, y)dx + G(x, y)dy.$$

Hurrah again! finally if $\vec{\boldsymbol{r}}$ is on the boundary $\mathcal{C}$ of S then (x, y) is on the boundary $\mathcal{C}_A$ of A and we have

$$\oint_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}} = \oint_{\mathcal{C}_A} F dx + G dy. \tag{96}$$

The results (95) and (96) reduce Stokes' theorem (86) to Green's theorem (84), thereby proving it in this particular case where S can be parametrized by x and y. ■

More general proof of Stokes' theorem

Indicial notation enables a fairly straightforward proof of Stokes' theorem for the more general case of a surface S in 3D space that can be parametrized by a 'good' function $\vec{\boldsymbol{r}}(s,t)$ (differentiable and integrability as needed). Such a surface S can fold and twist (it could even intersect itself!) and is therefore of a more general kind than those that can be parametrized by the cartesian coordinates x and y. The main restriction on S is that it must be 'orientable'. This means that it must have an 'up' and a 'down' as defined by the direction of the normal $\partial\vec{\boldsymbol{r}}/\partial s \times \partial\vec{\boldsymbol{r}}/\partial t$. The famous Mobius strip only has one side and is the classical example of a non-orientable surface. The boundary of the Mobius strip forms a knot.

► Let $x_i(s,t)$ represent the i component of the position vector $\vec{\boldsymbol{r}}(s,t)$ in the surface S with $i = 1, 2, 3$. Then

$$\begin{aligned}
(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot \left(\frac{\partial\vec{\boldsymbol{r}}}{\partial s} \times \frac{\partial\vec{\boldsymbol{r}}}{\partial t}\right) &= \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}\,\epsilon_{ilm}\frac{\partial x_l}{\partial s}\frac{\partial x_m}{\partial t} = \epsilon_{ijk}\epsilon_{ilm}\frac{\partial v_k}{\partial x_j}\frac{\partial x_l}{\partial s}\frac{\partial x_m}{\partial t} \\
&= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\frac{\partial v_k}{\partial x_j}\frac{\partial x_l}{\partial s}\frac{\partial x_m}{\partial t} \\
&= \frac{\partial v_k}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{\partial x_k}{\partial t} - \frac{\partial v_k}{\partial x_j}\frac{\partial x_k}{\partial s}\frac{\partial x_j}{\partial t} \\
&= \frac{\partial \bar{v}_k}{\partial s}\frac{\partial x_k}{\partial t} - \frac{\partial \bar{v}_k}{\partial t}\frac{\partial x_k}{\partial s} \\
&= \frac{\partial}{\partial s}\left(\bar{v}_k\frac{\partial x_k}{\partial t}\right) - \frac{\partial}{\partial t}\left(\bar{v}_k\frac{\partial x_k}{\partial s}\right) \\
&= \frac{\partial G(s,t)}{\partial s} - \frac{\partial F(s,t)}{\partial t},
\end{aligned} \tag{97}$$

where we have used (69), then the chain rule

$$\frac{\partial v_k}{\partial x_j}\frac{\partial x_j}{\partial s} = \frac{\partial v_k}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial v_k}{\partial x_2}\frac{\partial x_2}{\partial s} + \frac{\partial v_k}{\partial x_3}\frac{\partial x_3}{\partial s} = \frac{\partial \bar{v}_k}{\partial s},$$

and equality of mixed partials $\partial^2 x_k/\partial s\partial t = \partial^2 x_k/\partial t\partial s$ with $\bar{v}_k(s,t) = v_k\big(x_1(s,t), x_2(s,t), x_3(s,t)\big)$. The result (97) is equivalent to (94) with s, t in place of x, y. It is not only more general but also easier to obtain using indicial notation. That's because the cartesian parametrization actually breaks the symmetry of the problem. This proof demonstrates the power of the indicial notation and the summation convention. The first line of (97) is a *quintuple* sum over all values of the indices i, j, k, l and m! This would be unmanageable without the compact notation.

Now for any point on the surface S with position vector $\vec{\boldsymbol{r}}(s,t)$

$$\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}} = \bar{v}_i\left(\frac{\partial x_i}{\partial s}ds + \frac{\partial x_i}{\partial t}dt\right) = \left(\bar{v}_i\frac{\partial x_i}{\partial s}\right)ds + \left(\bar{v}_i\frac{\partial x_i}{\partial t}\right)dt = F(s,t)ds + G(s,t)dt, \tag{98}$$

and we have again reduced Stokes' theorem to Green's theorem (84) but expressed in terms of s and t instead of x and y. In details we have shown that

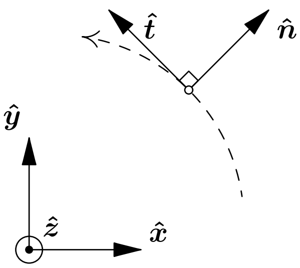
$$\int_S (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot d\vec{\boldsymbol{S}} = \int_A (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}) \cdot \left(\frac{\partial\vec{\boldsymbol{r}}}{\partial s} \times \frac{\partial\vec{\boldsymbol{r}}}{\partial t}\right) dsdt = \int_A \left(\frac{\partial G(s,t)}{\partial s} - \frac{\partial F(s,t)}{\partial t}\right) dsdt, \tag{99}$$

$$\oint_C \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{r}} = \oint_{C_A} F(s,t)ds + G(s,t)dt. \tag{100}$$

The right hand sides of (99) and (100) are equal by Green's theorem (84). ■

3.5 Divergence form of Green's theorem

The fundamental theorems (81) and (83) in $\mathbb{R}^2$ can be rewritten in a more palatable form.



The line element $d\vec{\boldsymbol{r}}$ at a point on the curve $\mathcal{C}$ is in the direction of the unit tangent $\hat{\boldsymbol{t}}$ at that point, so $d\vec{\boldsymbol{r}} = \hat{\boldsymbol{t}}\,ds$, where $\hat{\boldsymbol{t}}$ points in the counterclockwise direction of the curve. Then $\hat{\boldsymbol{t}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{n}}$ is the unit *outward* normal $\hat{\boldsymbol{n}}$ to the curve at that point and $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{t}}$ so

$$\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{t}} = \hat{\boldsymbol{x}} \cdot (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}) = (\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}}) \cdot \hat{\boldsymbol{n}} = -\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{n}} \tag{101}$$

$$\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{t}} = \hat{\boldsymbol{y}} \cdot (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{n}}) = (\hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}}) \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}}. \tag{102}$$

Hence since $d\vec{\boldsymbol{r}} = \hat{\boldsymbol{t}}\,ds$, the fundamental theorems (81) and (83) can be rewritten

$$\int_A \frac{\partial F}{\partial x}\,dA = \oint_{\mathcal{C}} F\hat{\boldsymbol{y}} \cdot d\vec{\boldsymbol{r}} = \oint_{\mathcal{C}} F\hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{n}}\,ds, \tag{103}$$

$$\int_A \frac{\partial F}{\partial y}\,dA = -\oint_{\mathcal{C}} F\hat{\boldsymbol{x}} \cdot d\vec{\boldsymbol{r}} = \oint_{\mathcal{C}} F\hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{n}}\,ds. \tag{104}$$

The right hand side of these equations is easier to remember since they have $\hat{\boldsymbol{x}}$ going with $\partial/\partial x$ and $\hat{\boldsymbol{y}}$ with $\partial/\partial y$ and both equations have positive signs. But there are hidden subtleties. The arclength element $ds = |d\vec{\boldsymbol{r}}|$ is positive by definition and $\hat{\boldsymbol{n}}$ must be the unit *outward* normal to the boundary, so if an explicit parametrization of the boundary curve is known, the bounds of integration should be picked so that $\oint_{\mathcal{C}} ds = \oint_{\mathcal{C}} |d\vec{\boldsymbol{r}}| > 0$ would be the length of the curve with a positive sign. For the $d\vec{\boldsymbol{r}}$ line integrals, the bounds of integration must correspond to counter-clockwise orientation of $\mathcal{C}$.
Formulas (103) and (104) can be combined in more useful forms. First, if u is the signed distance in the direction of the unit vector $\hat{\boldsymbol{u}}$, then the (directional) derivative in the direction $\hat{\boldsymbol{u}}$ is $\partial F/\partial u = \hat{\boldsymbol{u}} \cdot \vec{\boldsymbol{\nabla}} F \equiv u_x \partial F/\partial x + u_y \partial F/\partial y$, where $\hat{\boldsymbol{u}} = u_x\hat{\boldsymbol{x}} + u_y\hat{\boldsymbol{y}}$, therefore combining (103) and (104) accordingly we obtain

$$\boxed{\int_A \frac{\partial F}{\partial u}\,dA = \oint_{\mathcal{C}} F\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}}\,ds.} \tag{105}$$

This result is written in a coordinate-free form. It applies to *any* direction $\hat{\boldsymbol{u}}$ in the x,y plane.
Another useful combination is to add (103) to (104) written for a function $G(x, y)$ in place of $F(x, y)$ yielding the *divergence-form of Green's theorem*

$$\boxed{\int_A \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right)\,dA = \oint_{\mathcal{C}} (F\hat{\boldsymbol{x}} + G\hat{\boldsymbol{y}}) \cdot \hat{\boldsymbol{n}}\,ds.} \tag{106}$$

The left hand side integrand is easily recognized as the divergence $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$ of the vector field $\vec{\boldsymbol{v}}(x, y) = F\hat{\boldsymbol{x}} + G\hat{\boldsymbol{y}}$.

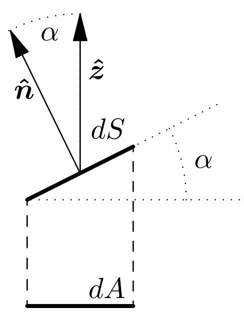
3.6 Gauss' theorem

Gauss' theorem is the 3D version of the divergence form of Green's theorem. It is proved by first extending the fundamental theorem of calculus to 3D.
If V is a bounded volume in 3D space and $F(x, y, z)$ is a scalar function of the cartesian coordinates (x, y, z), then we have

$$\int_V \frac{\partial F}{\partial z}\,dV = \oint_S F\,\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{n}} dS, \tag{107}$$

where S is the *closed* surface enclosing V and $\hat{\boldsymbol{n}}$ is the unit *outward* normal to S.
The proof of this result is similar to that for (81). Assume that the surface can be parametrized using x and y in two pieces: an upper 'hemisphere' at $z = z_u(x, y)$ and a lower 'hemisphere' at $z = z_l(x, y)$ with x, y in a domain A, the projection of S onto the x, y plane, that is the *same domain* for both the upper and lower surfaces. The closed surface S is not a sphere in general but we used the word 'hemisphere' to help visualize the problem. For a sphere, A is the equatorial disk, $z = z_u(x, y)$ is the northern hemisphere and $z = z_l(x, y)$ is the southern hemisphere.
Iterated integrals with $dV = dA\,dz$ and the fundamental theorem of calculus give

$$\int_V \frac{\partial F}{\partial z}\,dV = \int_A dA \int_{z_l(x,y)}^{z_u(x,y)} \frac{\partial F}{\partial z}\,dz = \int_A \big[F(x, y, z_u(x, y)) - F(x, y, z_l(x, y))\big]\,dA. \tag{108}$$



We can interpret that integral over A as an integral over the entire closed surface S that bounds V. All we need for that is a bit of geometry. If dA is the projection of the surface element $d\vec{\boldsymbol{S}} = \hat{\boldsymbol{n}}dS$ onto the x, y plane then we have $dA = \cos\alpha\,dS = \pm\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{n}}\,dS$. The + sign applies to the upper surface for which $\hat{\boldsymbol{n}}$ is pointing up (*i.e.* in the direction of $\hat{\boldsymbol{z}}$) and the - sign for the bottom surface where $\hat{\boldsymbol{n}}$ points down (and would be opposite to the $\hat{\boldsymbol{n}}$ on the side figure). Thus we obtain

$$\int_V \frac{\partial F}{\partial z}\,dV = \oint_S F\,\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{n}}\,dS, \tag{109}$$

where $\hat{\boldsymbol{n}}$ is the unit *outward* normal to S.
We can obtain similar results for the volume integrals of $\partial F/\partial x$ and $\partial F/\partial y$ and combine those to obtain

$$\boxed{\int_V \frac{\partial F}{\partial u}\,dV = \oint_S F\,\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{n}}\,dS,} \tag{110}$$

for arbitrary but fixed direction $\hat{\boldsymbol{u}}$. This is the 3D version of (105) and of the fundamental theorem of calculus.
We can combine this theorem into many useful forms. Writing it for $F(x, y, z)$ in the $\hat{\boldsymbol{x}}$ direction, with the $\hat{\boldsymbol{y}}$ version for a function $G(x, y, z)$ and the $\hat{\boldsymbol{z}}$ version for a function $H(x, y, z)$ we obtain *Gauss's theorem*

$$\boxed{\int_V \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}}\,dV = \oint_S \vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}\,dS,} \tag{111}$$

where $\vec{\boldsymbol{v}} = F\hat{\boldsymbol{x}} + G\hat{\boldsymbol{y}} + H\hat{\boldsymbol{z}}$. This is the 3D version of (106). Note that both (110) and (111) are expressed in coordinate-free forms. These are general results. The integral $\oint_S \vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}dS$ is the *flux* of $\vec{\boldsymbol{v}}$ through the surface S. If $\vec{\boldsymbol{v}}(\vec{\boldsymbol{r}})$ is the velocity of a fluid at point $\vec{\boldsymbol{r}}$ then that integral represents the time-rate at which volume of fluid flows through the surface S.
Gauss' theorem provides a coordinate-free interpretation for the divergence. Consider a small sphere of volume V and surface S centered at a point $\vec{\boldsymbol{r}}$, then Gauss' theorem states that

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}} = \lim_{V\to 0}\frac{1}{V}\oint_S \vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}\,dS. \tag{112}$$

Note that (110) and (111) are equivalent. We deduced (111) from (110), but we can also deduce (110) from (111) by considering the special $\vec{\boldsymbol{v}} = F\hat{\boldsymbol{u}}$ where $\hat{\boldsymbol{u}}$ is a unit vector independent of $\vec{\boldsymbol{r}}$. Then from our vector identities $\vec{\boldsymbol{\nabla}}\cdot(F\hat{\boldsymbol{u}}) = \hat{\boldsymbol{u}}\cdot\vec{\boldsymbol{\nabla}}F = \partial F/\partial u$.

3.7 Other useful forms of the fundamental theorem in 3D

A useful form of (110) is to write it in indicial form as

$$\boxed{\int_V \frac{\partial F}{\partial x_j}\, dV = \oint_S n_j F\, dS.} \tag{113}$$

Then with $f(\vec{\boldsymbol{r}})$ in place of $F(\vec{\boldsymbol{r}})$ we deduce that

$$\int_V \vec{\boldsymbol{e}}_j \frac{\partial f}{\partial x_j}\, dV = \oint_S \vec{\boldsymbol{e}}_j n_j f\, dS \tag{114}$$

since the cartesian unit vectors $\vec{\boldsymbol{e}}_j$ are independent of position. This result can be written in coordinate-free form as

$$\int_V \vec{\boldsymbol{\nabla}} f\, dV = \oint_S f\, \hat{\boldsymbol{n}}\, dS. \tag{115}$$

One application of this form of the fundamental theorem is to prove *Archimedes' principle.*
Next, writing (113) for v_k in place of F yields

$$\int_V \frac{\partial v_k}{\partial x_j}\, dV = \oint_S n_j\, v_k\, dS, \tag{116}$$

which can be multiplied by the position-independent ϵ_{ijk} and summed over all j and k to give

$$\int_V \epsilon_{ijk} \frac{\partial v_k}{\partial x_j}\, dV = \oint_S \epsilon_{ijk} n_j\, v_k\, dS. \tag{117}$$

The coordinate-free form of this is

$$\int_V \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}\, dV = \oint_S \hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}\, dS. \tag{118}$$

The integral theorems (115) and (118) provide yet other geometric interpretations for the gradient

$$\vec{\boldsymbol{\nabla}} f = \lim_{V\to 0} \frac{1}{V} \oint_S f\, \hat{\boldsymbol{n}}\, dS, \tag{119}$$

and the curl

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}} = \lim_{V\to 0} \frac{1}{V} \oint_S \hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}\, dS. \tag{120}$$

These are similar to the result (112) for the divergence. However, the geometric definition of the gradient as the vector pointing in the direction of greatest rate of change (sect. 2.1) and of the $\hat{\boldsymbol{n}}$ component of the curl as the limit of the local circulation per unit area as given by Stokes' theorem (87) are perhaps more fundamental.
In applications we use the fundamental theorem as we do in 1D, namely to reduce a 3D integral to a 2D integral, for instance. However we also use them the other way, to evaluate a complicated surface integral as a simpler volume integral for instance.

Exercises

1. If $\mathcal{C}$ is any closed curve in 3D space (i) calculate $\oint_{\mathcal{C}} \vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}}$ in two ways, (ii) calculate $\oint_{\mathcal{C}} \vec{\boldsymbol{\nabla}} f \cdot d\vec{\boldsymbol{r}}$ in two ways, where $f(\vec{\boldsymbol{r}})$ is a scalar function. [Hint: by direct calculation and by Stokes theorem].

2. If $\mathcal{C}$ is any closed curve in 3D space not passing through the origin calculate $\oint_{\mathcal{C}} r^{-3}\vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}}$ in two ways.

3. Calculate the circulation of the vector field $\vec{\boldsymbol{B}} = (\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}})/|\hat{\boldsymbol{z}} \times \vec{\boldsymbol{r}}|^2$ (i) about a circle of radius R centered at the origin in a plane perpendicular to $\hat{\boldsymbol{z}}$; (ii) about any closed curve $\mathcal{C}$ in 3D that does not go around the z-axis; (ii) about any closed curve $\mathcal{C}_0$ that does go around the $\hat{\boldsymbol{z}}$ axis. What's wrong with the z-axis anyway?

4. Consider $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ where $\vec{\boldsymbol{\omega}}$ is a constant vector, independent of $\vec{\boldsymbol{r}}$. (i) Evaluate the circulation of $\vec{\boldsymbol{v}}$ about the circle of radius R centered at the origin in the plane perpendicular to the direction $\hat{\boldsymbol{n}}$ by direct calculation of the line integral; (ii) Calculate the curl of $\vec{\boldsymbol{v}}$ using vector identities; (iii) calculate the circulation of $\vec{\boldsymbol{v}}$ about a circle of radius R centered at $\vec{\boldsymbol{r}}_0$ in the plane perpendicular to $\hat{\boldsymbol{n}}$.

5. If $\mathcal{C}$ is any closed curve in 2D, calculate $\oint_{\mathcal{C}} \hat{\boldsymbol{n}}\, dr$ where $\hat{\boldsymbol{n}}(\vec{\boldsymbol{r}})$ is the unit outside normal to $\mathcal{C}$ at the point $\vec{\boldsymbol{r}}$ of $\mathcal{C}$.

6. If S is any closed surface in 3D, calculate $\oint \hat{\boldsymbol{n}} dS$ where $\hat{\boldsymbol{n}}(\vec{\boldsymbol{r}})$ is the unit outside normal to S at a point $\vec{\boldsymbol{r}}$ on S.

7. If S is any closed surface in 3D, calculate $\oint p\, \hat{\boldsymbol{n}}\, dS$ where $\hat{\boldsymbol{n}}$ is the unit outside normal to S and $p(\vec{\boldsymbol{r}}) = (p_0 - \rho g \hat{\boldsymbol{z}} \cdot \vec{\boldsymbol{r}})$ where p_0, ρ and g are constants (This is *Archimedes' principle* with ρ as fluid density and g as the acceleration of gravity.)

8. Calculate the flux of $\vec{\boldsymbol{r}}$ through (i) the surface of a sphere of radius R centered at the origin in two ways; (ii) through the surface of the sphere of radius R centered at $\vec{\boldsymbol{r}}_0$; (iii) through the surface of a cube of side L with one corner at the origin in two ways.

9. (i) Calculate the flux of $\vec{\boldsymbol{v}} = \vec{\boldsymbol{r}}/r^3$ through the surface of a sphere of radius ϵ centered at the origin. (ii) Calculate the flux of that vector field through a closed surface that does not enclose the origin [Hint: use the divergence theorem] (iii) Calculate the flux through an arbitrary closed surface that encloses the origin [Hint: use divergence theorem and (i) to isolate the origin. What's wrong with the origin anyway?]

10. Calculate $\vec{\boldsymbol{\nabla}}|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|^{-1}$ and $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$ with $\vec{\boldsymbol{v}} = (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0)/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|^3$ where $\vec{\boldsymbol{r}}_0$ is a constant vector.

11. Calculate the flux of $\vec{\boldsymbol{v}}$ through the surface of a sphere of radius R centered at the origin when $\vec{\boldsymbol{v}} = \alpha_1(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_1)/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_1|^3 + \alpha_2(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_2)/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_2|^3$ where α_1 and α_2 are scalar constants and (i) $|\vec{\boldsymbol{r}}_1|$ and $|\vec{\boldsymbol{r}}_2|$ are both less than R; (ii) $|\vec{\boldsymbol{r}}_1| < R < |\vec{\boldsymbol{r}}_2|$. Generalize to $\vec{\boldsymbol{v}} = \sum_{i=1}^{N} \alpha_i(\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i)/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_i|^3$.

12. Calculate $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = \int_{V_0} \vec{\boldsymbol{f}} dV_0$ where V_0 is the inside of a sphere of radius R centered at O, $\vec{\boldsymbol{f}} = (\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0)/|\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_0|^3$ with $|\vec{\boldsymbol{r}}| > R$ and *the integral is over* $\vec{\boldsymbol{r}}_0$. Warning: this is essentially the gravity field or force at $\vec{\boldsymbol{r}}$ due to a sphere of uniform mass density. Supposedly, it took Newton *20 years* to figure it out... of course he was never taught calculus, he had to invent it, and vector calculus came much after him and he never knew about Gauss' theorem. The integral can be cranked out if you're good at analytic integration. But the smart solution is to realize that the integral over $\vec{\boldsymbol{r}}_0$ is essentially a sum over $\vec{\boldsymbol{r}}_i$ as in the previous exercise so we can figure out the flux of $\vec{\boldsymbol{F}}$ through any closed surface enclosing all of V_0. Now by symmetry $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}) = F(r)\hat{\boldsymbol{r}}$, so knowing the flux is enough to figure out $F(r)$. Newton would be impressed!