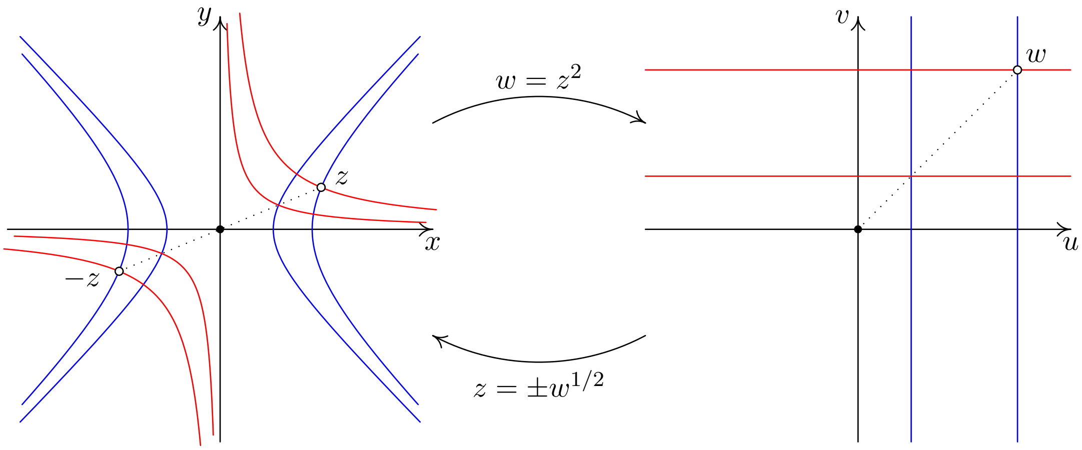
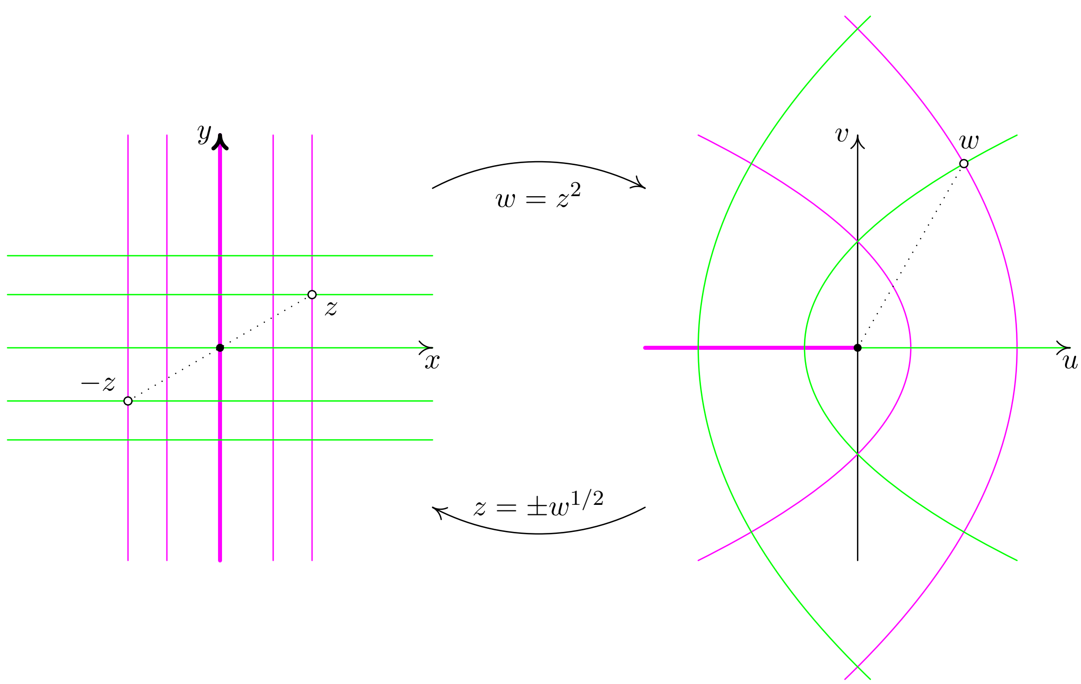
▶ $\boxed{w = z^2}$

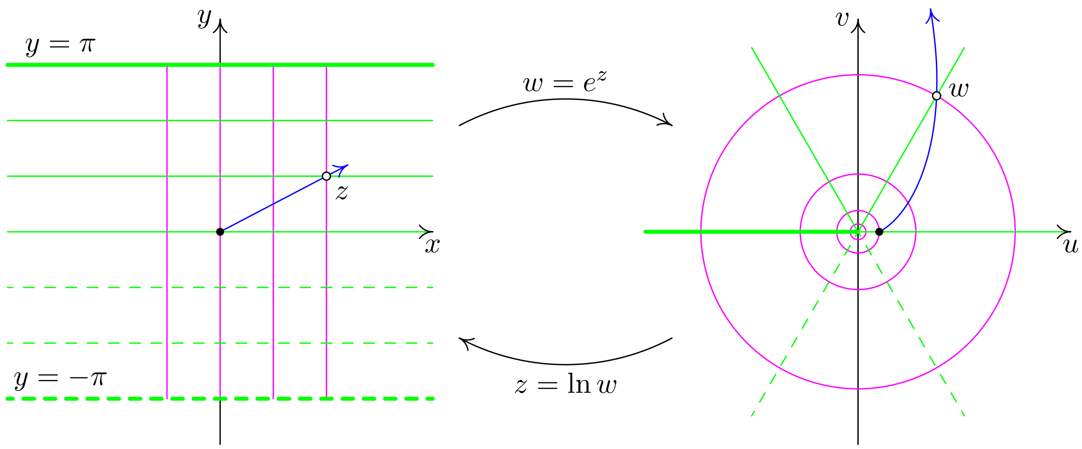
In cartesian forms $w = u + iv$, $z = x + iy$ so $u = x^2 - y^2$ and $v = 2xy$, with x, y, u and v all real.
First view: what do the $u(x, y)$ and $v(x, y)$ isocurves look like?
$u = x^2 - y^2 = u_0$, fixed, is a (blue) vertical line in the (u, v) plane and a (blue) hyperbola in the (x, y) plane. Likewise $v = 2xy = v_0$, fixed, is a (red) horizontal line in the (u, v) plane and a (red) hyperbola in the (x, y) plane. The blue and red curves intersect at 90 degrees in *both* planes. The dotted line intersects blue and red curves at 45 deg in *both* planes. All angles are preserved *except at* $z = 0$ where they are *doubled* in the w-plane.



Second view: what happens to (x, y) grid lines? *i.e.* $x(u, v)$ and $y(u, v)$ isocurves.
$u = x_0^2 - y^2$, $v = 2x_0 y$, x_0 fixed, is a (magenta) vertical line in (x, y) space and a (magenta) parabola in (u, v) plane. Likewise $u = x^2 - y_0^2$, $v = 2xy_0$, y_0 fixed, is a (green) horizontal line in the (x, y) plane and a (green) parabola in the (u, v) plane. The green and magenta curves intersect at 90 degrees in *both* planes. Angles between the dotted line and the green and magenta curves are the same in *both* planes, *except at* $z = w = 0$. What happens there?

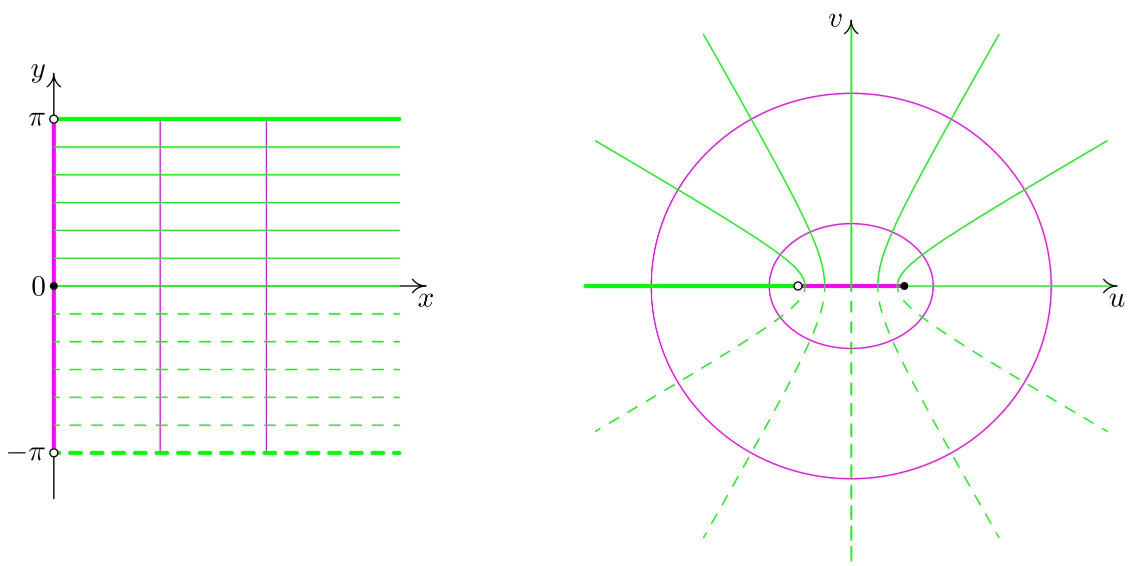


▶ $\boxed{w = e^z} = e^x e^{iy}$
Maps the *strip* $-\infty < x < \infty$, $-\pi < y \le \pi$ to the *entire* w-plane.
$z = x_0 + iy \to w = e^{x_0} e^{iy} \equiv$ circles of radius e^{x_0} in the w-plane (magenta).
$z = x + iy_0 \to w = e^x e^{iy_0} \equiv$ radial lines with polar angle $\arg(w) = y_0$ in w-plane (green).
$z = x + iax \to w = e^x e^{iax} \equiv$ radial lines out of the origin in z-plane mapped to *logarithmic spirals* in w-plane since $z = x + iax$ with a fixed (and real) becomes $w = e^x e^{iax} \equiv re^{i\theta}$ so $r = e^x$, $\theta = ax$ and $r = e^{\theta/a}$ in the w-plane (blue).



Notes: $z = 0 \to w = 1$. $e^{z+2i\pi} = e^z$, periodic of complex period $2i\pi$, so e^z maps an infinite number of z's to the same w. The inverse function $z = \ln w = \ln|w| + i\arg(w)$ showed in this picture corresponds to the definition $-\pi < \arg(w) \le \pi$. All angles are preserved *e.g.* the angles between green and magenta curves, as well as between blue and colored curves, except at $w = 0$. What z's correspond to $w = 0$?

▶ $\boxed{w = \cosh(z)} = (e^z + e^{-z})/2 = (e^x e^{iy} + e^{-x} e^{-iy})/2 = \cosh x \cos y + i \sinh x \sin y \equiv u + iv$.
Maps the *semi-infinite strip* $0 \le x < \infty$, $-\pi < y \le \pi$ to the *entire* w-plane.
$\cosh(z) = \cosh(-z)$ and $\cosh(z + 2i\pi) = \cosh(z)$, even in z and periodic of period $2i\pi$.
$x = x_0 \ge 0 \to u = \cosh x_0 \cos y$, $v = \sinh x_0 \sin y$, $\equiv$ *ellipses* in the w-plane (magenta).
$y = y_0 \ge 0 \to u = \cosh x \cos y_0$, $v = \sinh x \sin y_0$, $\equiv$ *hyperbolas* in the w-plane (green).
This mapping gives *orthogonal, confocal elliptic coordinates.*



What happens to angles at $z = 0, \pm i\pi$? Show that the inverse map is $z = \ln(w + \sqrt{w^2 - 1})$, but for what definition of $\sqrt{w^2 - 1}$? (*not Matlab's!*) The line from $w = -\infty$ to $w = 1$ is a *branch cut*, our definition for $\ln(w + \sqrt{w^2 - 1})$ is discontinuous across that line.