

Calculate gravitational or electrical force induced by a hollow sphere $a^2 < x^2 + y^2 + z^2 < b^2$. (Greenberg p.755.)

$$I \equiv \iiint \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - z_0)\hat{\mathbf{k}}}{(x^2 + y^2 + (z - z_0)^2)^{3/2}} dV$$

by symmetry

$$I = \hat{\mathbf{k}} \iiint \frac{(z - z_0)}{(x^2 + y^2 + (z - z_0)^2)^{3/2}} dV$$

Use spherical coordinates: $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$, then $dV = \rho^2 \sin \phi d\rho d\phi d\theta$.

$$I = 2\pi \hat{\mathbf{k}} \int_a^b \int_0^\pi \frac{\rho \cos \phi - z_0}{(\rho^2 - 2\rho z_0 \cos \phi + z_0^2)^{3/2}} \rho^2 \sin \phi d\rho d\phi$$

Let $u = (\rho^2 - 2\rho z_0 \cos \phi + z_0^2)$ then $du = 2z_0 \rho \sin \phi d\phi$ and

$$\begin{aligned} I &= \frac{2\pi \hat{\mathbf{k}}}{4z_0^2} \int_a^b \int_{(\rho-z_0)^2}^{(\rho+z_0)^2} \frac{\rho^2 + z_0^2 - u}{u^{3/2}} \rho d\rho du \\ &= \frac{\pi \hat{\mathbf{k}}}{z_0^2} \int_a^b \left\{ (\rho^2 - z_0^2) \left(\frac{1}{|\rho - z_0|} - \frac{1}{|\rho + z_0|} \right) + (|\rho - z_0| - |\rho + z_0|) \right\} \rho d\rho \\ &= \frac{\pi \hat{\mathbf{k}}}{z_0^2} \int_a^b \{ (\text{sign}(\rho - z_0)(\rho + z_0) + z_0 - \rho) + (|\rho - z_0| - \rho - z_0) \} \rho d\rho \end{aligned}$$

Without loss of generality, assume $z_0 > 0$ (can find answer for $z_0 < 0$ by symmetry), then $|\rho + z_0| = \rho + z_0$.

If $\rho < z_0$ then $|\rho - z_0| = z_0 - \rho$ and

$$I = \frac{\pi \hat{\mathbf{k}}}{z_0^2} \int_a^b -4\rho^2 d\rho = -\frac{4\pi}{3}(b^3 - a^3) \frac{\hat{\mathbf{k}}}{z_0^2} = -\frac{M\hat{\mathbf{k}}}{z_0^2}$$

If $\rho > z_0$ then $|\rho - z_0| = \rho - z_0$ and $I = 0$.