1. [25pts] *Consider the vector field* $\mathbf{F} = y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}} + \sqrt{5}\,\hat{\mathbf{z}}$. *Calculate (a) its divergence, (b) its curl, (c) its flux through the surface* $S \equiv z = 4 - x^2 - y^2 \geq 0$ *in the* $\hat{\mathbf{z}}$ *direction. Justify your calculations.*

(a), (b): $\nabla \cdot \mathbf{F} = 0$, $\nabla \times \mathbf{F} = 0$. (c) The surface S is a paraboloid from $z = 0$ to its tip at $z = 4$. This is not a closed surface. The flux is $\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$, where $\hat{\mathbf{n}}$ is in the direction of $\hat{\mathbf{z}}$, thus $\hat{\mathbf{z}} \cdot \hat{\mathbf{n}} \geq 0$ but $\hat{\mathbf{n}} \neq \hat{\mathbf{z}}$!. Flux $\Rightarrow$ think: **Divergence Theorem**

$$\int_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_V \nabla \cdot \mathbf{F}\, dV$$

where $\hat{\mathbf{n}}$ is the **outward** unit normal to the surface and ∂V is the **closed** surface that forms the boundary of V, $\partial V \equiv S + A$ where A is the disk $z = 0$, $x^2 + y^2 \leq 4$. From the divergence theorem:

$$\int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS + \int_A \mathbf{F} \cdot (-\hat{\mathbf{z}})\, dA = 0 \Rightarrow \int_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_A \mathbf{F} \cdot \hat{\mathbf{z}}\, dA = 4\pi\sqrt{5}.$$

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2. [25pts] *Sketch the curve* $\mathcal{C}$ *given by* $x = \cos t$, $y = \sin t$, $z = \sin t$ *with* $0 \leq t \leq 2\pi$, *then calculate*

$$\int_C 2xe^{2y} dx + (2x^2 e^{2y} + 2y \tan z) dy + \frac{y^2}{\cos^2 z} dz.$$

Justify your calculations.
This curve is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $z = y$. It is an ellipse in the $y = z$ plane. Its projection in the $z = 0$ plane is a unit circle. This is a **closed** curve. The integral is a line integral over a closed curve $\oint_C \mathbf{F} \cdot \mathbf{ds} \Rightarrow$ think **Stokes Theorem**, think **curl**, verify $\nabla \times \mathbf{F} = 0$, thus $\mathbf{F}$ is conservative $\Leftrightarrow$ $\mathbf{F} = \nabla\phi$. Then by Stokes or $\oint_C \mathbf{F} \cdot \mathbf{ds} = \oint_C \nabla\phi \cdot \mathbf{ds} = \oint_C d\phi = 0$.

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3. [30pts] *The region inside the triangle* $(0,0)$, $(1,0)$ *and* $(0,i)$ *in the complex* z*-plane is mapped into the* w*-plane. Sketch the image region in the* w*-plane and show explicitly where the various vertices and edges have been mapped. Specify angles and shapes of edge images (e.g. straight line, circle, ellipse, hyperbola,...). Do so for each of the three following maps:*

$$\text{(a) } w = 1/z, \qquad \text{(b) } w = e^z, \qquad \text{(c) } w = (z-1)^2.$$

Give a name to vertices, e.g. $A \equiv (0,0)$, $B \equiv (1,0)$, $C \equiv (0,i)$.
Edge AB: $0 \leq x \leq 1$, $y = 0$ or $z = x$. **Edge** AC: $x = 0$, $0 \leq y \leq 1$ or $z = iy$.
Edge BC: $0 \leq x \leq 1$, $y = 1 - x$ or $z = x + i(1 - x)$.

(a) $w = 1/z$:
AB: $w = 1/z = 1/x$, real line from 1 to $+\infty$. B mapped to $w = 1$, A mapped to infinity.
AC: $w = 1/(iy) = -i/y$, imaginary line from $-i$ to $-i\infty$. C mapped to $w = -i$, A mapped to $-i\infty$.
BC: not so obvious. Let's consider another point $D \equiv z = e^{i\pi/4}/\sqrt{2}$ which is midpoint between B and C. D is mapped to $w = \sqrt{2}e^{-i\pi/4} = 1 - i$. The mapping $w = 1/z$ is nice as long as we're away from 0. So the image of the line BDC should be "nice" (no corners in particular). What nice curve could go through $w = 1$, $1 - i$ and $-i$? (plot those points) Lots, of course, but let's stick to simple curves first, Circle of center $(1 - i)/2$ radius $1/\sqrt{2}$?
Yes, but you need to know $e^{i\theta} = \cos\theta + i \sin\theta$ inside and out to show it! Use Polar form: $z = re^{i\theta}$, $x = r\cos\theta$, $y = r\sin\theta$, but $y = 1 - x = 1 - r\cos\theta$, so $r = 1/(\cos\theta + \sin\theta)$, thus BC in polar is $z = e^{i\theta}/(\cos\theta + \sin\theta)$. Now $\cos\theta + \sin\theta = (e^{i\theta} + e^{-i\theta})/2 + (e^{i\theta} - e^{-i\theta})/(2i) = [(1+i)e^{i\theta} - (1-i)e^{-i\theta}]/(2i)$

but $1+i=\sqrt{2}e^{i\pi/4}$ and $1-i=\sqrt{2}e^{-i\pi/4}$, so $\cos\theta+\sin\theta=\sqrt{2}\left(e^{i(\theta+\pi/4)}-e^{-i(\theta+\pi/4)}\right)/(2i)=\sqrt{2}\sin(\theta+\pi/4)$.
The image of BC is then

$$w=\frac{1}{z}=\frac{\sqrt{2}}{2i}\left(e^{i(\theta+\pi/4)}-e^{-i(\theta+\pi/4)}\right)e^{-i\theta}=\frac{1}{\sqrt{2}}e^{-i\pi/4}+\frac{1}{\sqrt{2}}e^{-i(2\theta-\pi/4)}$$

(using $i=e^{i\pi/2}$) and thus $|w-e^{-i\pi/4}/\sqrt{2}|=1/\sqrt{2}$, Circle of center $(1-i)/2$ radius $1/\sqrt{2}$.

(b) $w=e^z$:
AB: $w=e^x$, real line from 1 to e. AC: $w=e^{iy}$, circular arc from $w=1$ to $w=e^i$. BC: $w=e^x e^{i(1-x)}\Rightarrow|w|=e^x$, $\arg(w)=1-x\rightarrow|w|=ee^{-\arg(w)}$, equiangular spiral [see "course related links" on web page] from $w=e$ to $w=e^i$.

(c) $w=(z-1)^2$:
AB: $w=(x-1)^2$, real line from 1 to 0. AC: $w=(iy-1)^2=1-y^2-2iy$, parabola from $w=1$ to $w=-2i$. BC: $w=(x+i(1-x)-1)^2=2i(x-1)$, imaginary line from $w=0$ to $w=-2i$.

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4. [20pts] *The equation $F(x,y)=0$ represents a curve $\mathcal{C}$ in a 2D plane. We would like to know the signed distance function $d(x,y)$ that represents the distance from the point (x,y) to the curve $\mathcal{C}$ ["signed" means that $d>0$ on one side of $\mathcal{C}$ and $d<0$ on the other side].*
(a) Give an explicit example of a curve $\mathcal{C}$ and the associated signed distance function $d(x,y)$,
(b) find an equation for the function $d(x,y)$,
(c) specify all that is known about $d(x,y)$ along the curve $\mathcal{C}$.
[Hint: what is the direction of fastest increase of $d(x,y)$? what is the rate of increase of $d(x,y)$ in that direction?]

(a) $x=0$ (vertical y-axis) and $d(x,y)=x$. Other example: $x-y=0$ (diagonal through origin) and $d(x,y)=(x-y)/\sqrt{2}$. Yet another: $x^2+y^2-4=0$ (circle of radius 2) and $d(x,y)=sign(x^2+y^2-4)\sqrt{|x^2+y^2-4|}$.
(b) From hint: $d(x,y)$ increases fastest in direction of its gradient, i.e. $\nabla d/|\nabla d|$. The rate of increase of d in that direction is $\nabla d\cdot\nabla d/|\nabla d|$. This represents the rate of increase of d with respect to the *distance* in the direction of the gradient of d, that's the rate of change of d with respect to d itself! so we have

$$\frac{\nabla d}{|\nabla d|}\cdot\nabla d=1,\quad\text{or}\quad|\nabla d|=1,\quad\text{or}\quad d_x^2+d_y^2=1.$$

This is the governing equation for $d(x,y)$. It is called the *eikonal equation.* It comes up in the *level-set method* which is a fairly recent numerical technique to track fronts (multiphase, combustion, ...) and in wave propagation problems where its contours correspond to wave fronts. It is a first-order nonlinear Partial Differential Equation (PDE).
(c) Along the curve $\mathcal{C}\equiv F(x,y)=0$ we have $d(x,y)=0$ of course and $\nabla d=\nabla F/|\nabla F|$.