YOUR NAME:

1. The equation of state of a not-so-perfect gas is

$$p\,(v-b) = RT$$

where p, v and T are the pressure, specific volume and temperature of the gas, respectively, while b and R are constants. If v_0 is the specific volume corresponding to pressure p_0 and temperature T_0, give an explicit formula for the specific volume v_1 that corresponds to p_1 and T_1 by using a linear approximation (i.e. assuming that p_1 and T_1 are "close" to p_0 and T_0 respectively. **For extra credit**, specify how you would determine how close is close enough for the linear approximation to be valid).

Taylor Series:

$$v_1 \approx v_0 + \left(\frac{\partial v}{\partial T}\right)_0 (T_1 - T_0) + \left(\frac{\partial v}{\partial p}\right)_0 (p_1 - p_0)$$

Here it is easy enough to express $v = v(p,T)$ explicitly, otherwise in general use implicit form if $f(p,v,T) = 0$ then

$$\frac{\partial v}{\partial T} = -\frac{\partial f}{\partial T} / \frac{\partial f}{\partial v}$$

and likewise for $\partial v/\partial p$.
To determine whether this is a valid approximation you can compare the linear terms to the quadratic terms that appear at the next order in the Taylor series.

2. Find the points on the curve $x - y^2 + 2 = 0$ that are closest to the origin.

Minimize $F(x,y) = x^2 + y^2$ ("objective function") subject to $G(x,y) = x - y^2 + 2 = 0$ (constraint). Use Lagrange multipliers $\Rightarrow$ minimize $H(x,y,\lambda) = (x^2+y^2) - \lambda(x - y^2 + 2)$.

$$\frac{\partial H}{\partial x} = 0 = 2x - \lambda \qquad (1)$$

$$\frac{\partial H}{\partial y} = 0 = 2y + 2\lambda y \qquad (2)$$

$$\frac{\partial H}{\partial \lambda} = 0 = x - y^2 + 2 \qquad (3)$$

Hence, $y = 0 \Rightarrow x = -2$ or $\lambda = -1 \Rightarrow x = -1/2 \Rightarrow y = \pm\sqrt{3/2}$.
Evaluating $x^2 + y^2$ at these 3 solutions (and a simple plot of $x - y^2 + 2$) shows that the first one is a local max and the other two are two local mins. There are no other extremas.

3. Find an equation for the plane containing the three points $P = (1,4,3)$, $Q = (2,0,-1)$ and $R = (0,0,5)$. What is the area of the triangle PQR? Give a parametric representation of the straight line through the points P and Q. Can you also find a parametric representation for the plane PQR in the form $x = x(u,v)$, $y = y(u,v)$ and $z = z(u,v)$?

The vectors $\vec{PQ} = \vec{OQ} - \vec{OP} = (1,-4,-4)$ and $\vec{QR} = \vec{OR} - \vec{OQ} = (-2,0,6)$ (for instance) are in the plane (the point O is the origin of the system of coordinates). So $\vec{N} = \vec{PQ} \times \vec{QR} = (-24,2,-8)$ is normal to the plane. If (x,y,z) is a point in the PQR plane, then the vector joining that point to any other one on the plane, R say, is perpendicular to $\vec{N}$:

$$(x,y,z-5)\cdot\vec{N} = 0 \quad \text{or} \quad 12x - y + 4z = 20.$$

The area of the triangle PQR is 1/2 the norm of the cross-product, $\vec{N}$: $Area = \sqrt{24^2 + 2^2 + 8^2}\,/2 = \sqrt{161}$.
A straight line through P and Q has the parametric representation $\vec{OX} = \vec{OP} + t\vec{PQ}$ or

$$(x, y, z) = (1, 4, 3) + t(1, -4, -4),$$

where t is the parameter.
A vector $\vec{OX}$ in the plane has the parametric representation $\vec{OX} = \vec{OP} + u\vec{PQ} + v\vec{QR}$ or

$$(x, y, z) = (1, 4, 3) + u(1, -4, -4) + v(-2, 0, 6),$$

where u and v are the real parameters. These parametric representations are clearly not unique.

4. A bead is constrained to move along a straight rod that rotates around the z-axis at constant angular velocity Ω and is attached at the origin. The angle between the rod and the z-axis is constant. If the position vector of the bead with respect to the origin is $\mathbf{R} = s\mathbf{e}$, where $\mathbf{e}$ is a unit vector in the direction of the rod, express the velocity of the bead in terms of $\dot{s}$, Ω, $\hat{\mathbf{k}}$ and $\mathbf{e}$.

$$\frac{d}{dt}\mathbf{R} = \dot{s}\mathbf{e} + s\dot{\mathbf{e}}$$

what is $\dot{\mathbf{e}}$? $\mathbf{e}$ is a unit vector so $\dot{\mathbf{e}} \cdot \mathbf{e} = 0$. The angle between $\hat{\mathbf{k}}$ and $\mathbf{e}$ is a constant, so these two vectors together with $\hat{\mathbf{k}} \times \mathbf{e}$ form a rigid frame that rotates with angular velocity vector $\Omega\hat{\mathbf{k}}$. Thus

$$\dot{\mathbf{e}} = \Omega\hat{\mathbf{k}} \times \mathbf{e},$$

and

$$\frac{d}{dt}\mathbf{R} = \dot{s}\mathbf{e} + s\Omega\hat{\mathbf{k}} \times \mathbf{e}.$$

5. A wheel is rolling along a plane. Find a parametrization for the trajectory of a point on its rim. What is the distance travelled by such a point during one complete revolution of the wheel?

Look back in your favorite Calculus book. The parametrization consists of a circle plus a translation. A parametrization for a circle is $x = R\cos\theta$, $y = R\sin\theta$ where θ is an angle measured from the center. During a rotation by an angle θ a point initially at $x = 1$, $y = 0$ is now at $x = R\cos\theta$, $y = R\sin\theta$ with respect to the center. But the wheel has moved by a distance $R\theta$, the arclength. So with respect to a frame of reference fixed at the original center the coordinates of the point that was at $x = 1$, $y = 0$ initially are now

$$x = R(\theta + \cos\theta), \quad y = R\sin\theta.$$

Distance travelled by that point over one complete revolution:

$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int_0^{2\pi} R\sqrt{(1 - \sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} R\sqrt{2 - 2\sin\theta}\, d\theta = R\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = R\int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = 8R.$$

6.

$$\int_0^4 \left(\int_{y/2}^{\sqrt{y}} f(x, y)dx\right) dy = \int_0^2 \left(\int_{x^2}^{2x} f(x, y)dy\right) dx$$

(a plot helps a lot).