

And God said...

$$\epsilon_0 \oint \oint \vec{E} \cdot \vec{n} dS = \iiint q dV$$

$$\oint \vec{B} \cdot d\vec{R} = \mu_0 \iint \vec{J} \cdot \vec{n} dS + \mu_0 \epsilon_0 \frac{d}{dt} \iint \vec{E} \cdot \vec{n} dS$$

$$\oint \vec{E} \cdot d\vec{R} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{n} dS$$

$$\oint \oint \vec{B} \cdot \vec{n} dS = 0$$

(The integral forms of the laws of electromagnetism: Maxwell's equations. It is those integral forms that one would verify experimentally.) $\vec{E}$ is the electric field, q is the charge density, $\vec{B}$ is the magnetic induction field, $\vec{J}$ is the electric current, ϵ_0 and μ_0 are the electric and magnetic permeability of vacuum, respectively.

... but then God took Math 321 and learned:

The DIVERGENCE (GAUSS) Theorem:

$$\oint \oint \vec{F} \cdot \vec{n} dS = \iiint \vec{\nabla} \cdot \vec{F} dV$$

The STOKES Theorem:

$$\oint \vec{F} \cdot d\vec{R} = \iint (\vec{\nabla} \times \vec{F}) \cdot \vec{n} dS$$

... and God re-wrote the equations as

$$\epsilon_0 \iiint \vec{\nabla} \cdot \vec{E} dV = \iiint q dV$$

$$\iint (\vec{\nabla} \times \vec{B}) \cdot \vec{n} dS = \mu_0 \iint \vec{J} \cdot \vec{n} dS + \mu_0 \epsilon_0 \frac{d}{dt} \iint \vec{E} \cdot \vec{n} dS$$

$$\iint (\vec{\nabla} \times \vec{E}) \cdot \vec{n} dS = -\frac{d}{dt} \iint \vec{B} \cdot \vec{n} dS$$

$$\iiint \vec{\nabla} \cdot \vec{B} dV = 0$$

... but then God realized that these equations held for *any* closed volume V and surface S , which meant that the integrand inside the integrals had to be equal to each other pointwise, so

$$\begin{aligned}\epsilon_0 \vec{\nabla} \cdot \vec{E} &= q \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B} \\ \vec{\nabla} \cdot \vec{B} &= 0\end{aligned}$$

then in a vacuum when $\vec{J} = q = 0$ taking the curl of the third equation and using the second equation yields

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} = -\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{E}$$

but using the first equation it turns out that $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E}$ and so God found

$$\frac{\partial^2}{\partial t^2} \vec{E} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \vec{E} \tag{*}$$

... and there was light

(Equation (*) is the *wave equation*, it has traveling wave solutions of the form $e^{i(x-ct)}$ where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light. However, God needed to take **Math 322** to learn that. God later told all this to *Sir James Clerk Maxwell* who stated the equations in 1864.)

Electrostatics $\equiv \partial/\partial t = 0$ and $\vec{B} = \vec{J} = 0$ then Maxwell's equations reduce to

$$\begin{aligned}\epsilon_0 \vec{\nabla} \cdot \vec{E} &= q \\ \vec{\nabla} \times \vec{E} &= 0\end{aligned}$$

In electrostatics the Electric field is curl-free (*irrotational*). Hence, it can be written as the gradient of a scalar field: $\vec{E} = -\vec{\nabla}V$. V is the electric potential. The first equation then yields:

$$\nabla^2 V = -q/\epsilon_0$$

This is *Poisson's equation*. If $q = 0$, then we get *Laplace's equation* $\nabla^2 V = 0$.

Magnetostatics

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} \\ \vec{\nabla} \cdot \vec{B} &= 0\end{aligned}$$

The magnetic field is divergence-free (*solenoidal*). (This is true in electrodynamics as long as there are no *magnetic monopoles*. Experiments have shown that the existence of magnetic monopoles is highly unlikely). Hence, the magnetic field can be written as the curl of some other field (the magnetic potential): $\vec{B} = \vec{\nabla} \times \vec{\psi}$, which is defined to within an arbitrary gradient as $\vec{\psi}$ and $\vec{\psi} + \vec{\nabla}\varphi$ have the same curl. To make ψ definite, one can set $\vec{\nabla} \cdot \vec{\psi} = 0$ then $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ implies

$$\nabla^2 \vec{\psi} = -\mu_0 \vec{J}$$

because $\vec{\nabla} \times (\vec{\nabla} \times \vec{\psi}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\psi}) - \nabla^2 \vec{\psi}$. So the magnetic potential also satisfies a Poisson equation.

2D, irrotational, incompressible Fluid Dynamics

Irrotationality is the strongest assumption. No real fluid behaves like this but there are a few important problems for which this gives a good approximation to the real fluid flow, motion due to surface waves or bubble oscillations and flow around an airfoil, for instance. Let $\vec{v}$ be the velocity field. Incompressibility means $\vec{\nabla} \cdot \vec{v} = 0$. This is a consequence of conservation of mass and the divergence theorem. Irrotationality means $\vec{\nabla} \times \vec{v} = 0$. As mentioned above, irrotationality is more delicate and follows from conservation of vorticity ($\equiv \vec{\nabla} \times \vec{v}$) in inviscid flow. So if there is no vorticity initially, then the vorticity remains zero for all times in inviscid flows (problem is no fluid is really inviscid...).

In any case, the velocity field of incompressible, irrotational flows is both divergence-free (solenoidal) and curl-free (irrotational). So $\vec{v}$ can be written as the curl of a vector field (the *streamfunction*) or as the gradient of a scalar (the *potential*).

In 2D: $\vec{v} = v_1 \hat{i} + v_2 \hat{j}$ with

$$\begin{aligned}\vec{\nabla} \cdot \vec{v} &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \\ \vec{\nabla} \times \vec{v} &= \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \hat{k} = 0\end{aligned}$$

▷ to automatically satisfy incompressibility we can use a streamfunction ψ and write

$$v_1 = \frac{\partial \psi}{\partial y} \quad v_2 = -\frac{\partial \psi}{\partial x}$$

then irrotationality implies

$$\nabla^2 \psi = 0.$$

▷ to automatically satisfy irrotationality we can use a potential ϕ and write

$$v_1 = \frac{\partial \phi}{\partial x} \quad v_2 = \frac{\partial \phi}{\partial y}$$

then incompressibility implies

$$\nabla^2 \phi = 0.$$

So both the streamfunction ψ and the potential ϕ satisfy Laplace's equation. However the boundary conditions are quite different. The velocity is the gradient of the potential so the velocity is perpendicular to equipotential lines. In contrast, the velocity is perpendicular to the gradient of the streamlines (*i.e.* contour lines of equal ψ), thus it is parallel to the streamlines (hence the names “streamline” and “streamfunction”).

Example: Flow in the 90-degree wedge $x, y \geq 0$. The boundary conditions are that there is no flow through the walls $x = 0$ and $y = 0$, so

$$v_1 = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \quad \text{on } x = 0$$

$$v_2 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = 0 \quad \text{on } y = 0$$

So ψ is constant along the boundary (the boundary is a streamline) while equipotentials are perpendicular to the boundaries. The solutions are $\phi = x^2 - y^2$ and $\psi = 2xy$.

• Note that the definitions

$$v_1 = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$v_2 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

imply that ϕ and ψ satisfy the **Cauchy-Riemann** equations. Hence, the complex function $\phi(x, y) + i\psi(x, y)$ is guaranteed to be an analytic function of $z = x + iy$

$$\phi(x, y) + i\psi(x, y) = f(z)$$

and the velocity is given by $f'(z) = \partial f / \partial x = v_1 - i v_2$.