Problem 5 on exam 4 was to calculate $\oint_{\mathcal{C}} f(z)\,dz$ where $\mathcal{C}$ is the circle of radius 2 centered at $z = 0$ and

$$f(z) = \frac{2}{z^2 - 4z + 3} = \frac{1}{z-3} - \frac{1}{z-1}.$$

Jared Brill asked: *"what's wrong with the following solution:"* Parametrize the circle by $z = 2e^{it}$, then $dz = 2ie^{it}dt$ with $t = 0 \to 2\pi$ and the integrals become

$$\oint_{\mathcal{C}} \left(\frac{1}{z-3} - \frac{1}{z-1}\right) dz = \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}-3}dt - \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}-1}dt. \tag{1}$$

Now make the change of variable $u = 2e^{it} - 3$ in the first and $v = 2e^{it} - 1$ in the second, so $du = 2ie^{it}dt$ and $dv = 2ie^{it}dt$, and $t = (0, 2\pi) \Rightarrow u = (-1, -1)$, while $t = (0, 2\pi) \Rightarrow v = (1, 1)$. So the two integrals become

$$= \int_{-1}^{-1} \frac{1}{u}\,du - \int_1^1 \frac{1}{v}\,dv, \tag{2}$$

and the result looks to be zero since the upper and lower bounds in both integrals are identical! Hmm ... Good question! what's going on here?!
Note first that we could get to (2) faster by skipping all the $z = 2e^{it}$ thing.
Rewrite the integral over the circle of radius 2 as an integral from $z = 2$ to ... $z = 2$!

$$\oint_{\mathcal{C}} \left(\frac{1}{z-3} - \frac{1}{z-1}\right) dz \text{ “=” } \int_2^2 \left(\frac{1}{z-3} - \frac{1}{z-1}\right) dz \tag{3}$$

now let $u = z - 3$ for the first integral and $v = z - 1$ for the second to obtain (2).
This goes back to the basics: every complex integral is an integral over a curve in the complex plane. The result depends on the path in general. Writing $\int_2^2$, we throw away all path information. Not good, no wonder we're getting lost.

The solution to this problem is to use Cauchy's theorem to 'shrink wrap' the contour of integration around the 'poles'. We did this in class for the general $\oint_{\mathcal{C}}(z-a)^n dz$ which we found to be $2\pi i$ when a is inside $\mathcal{C}$, AND $n = -1$. All other cases gave 0. That's box (67) in the complex notes. Here $z = 3$ is outside the circle of radius 2 and $z = 1$ is inside so

$$\oint_{|z|=2} \frac{1}{z-3}\,dz = 0 \quad \text{but} \quad \oint_{|z|=2} \frac{1}{z-1}\,dz = 2\pi i. \tag{4}$$

While we're at it, let's look at the integral of $(z-1)^{-1}$ over the circle of radius 2 in various ways. By Cauchy's theorem we can shrink wrap the contour around the pole at $z = 1$, that is

$$\oint_{|z|=2} \frac{1}{z-1}\,dz = \oint_{|z-1|=\epsilon} \frac{1}{z-1}\,dz \tag{5}$$

with ϵ as small as we want (but not zero). The integrand is the same but the contours are different. We're substituting contours, not variables.
Let's parametrize both of these integrals. For the first we have $z = 2e^{it}$ and for the 2nd we have $z = 1 + \epsilon e^{it}$, so

$$\oint\limits_{|z|=2} \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}-1} dt = ?!$$

$$= \oint\limits_{|z-1|=\epsilon} \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{i\epsilon e^{it}}{\epsilon e^{it}} dt = i\int_0^{2\pi} dt = 2\pi i. \tag{6}$$

Et voilà! the first t-integral is hard, the 2nd is easy. Shrink wrapping is good!
Digesting the first integral some more:

$$\oint\limits_{|z|=2} \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}-1} dt$$

$$= \int_0^{2\pi} \frac{2ie^{it}}{2e^{it}-1}\,\frac{2e^{-it}-1}{2e^{-it}-1} dt = \int_0^{2\pi} \frac{4i-2ie^{it}}{4-2e^{it}-2e^{-it}+1} dt = \tag{7}$$

$$= \int_0^{2\pi} \frac{2\sin t + i(4-2\cos t)}{5-4\cos t} dt = \int_0^{2\pi} \frac{2\sin t}{5-4\cos t} dt + i\int_0^{2\pi} \frac{4-2\cos t}{5-4\cos t} dt$$

By Cauchy's theorem we know that these two complicated looking t-integrals evaluate to 0 and 2π, respectively.

If we did not know that they are the real and imaginary part of $(z-1)^{-1}$ about the circle $|z| = 2$, we could unwrap them by the complex change of variables $z = e^{it}$, with $dz = ie^{it}dt$, so $dt = dz/(iz)$, and $\cos t = (e^{it} + e^{-it})/2 = (z + 1/z)/2$ giving

$$\int_0^{2\pi} \frac{4-2\cos t}{5-4\cos t} dt = \oint\limits_{|z|=1} \frac{4-z-1/z}{5-2z-2/z}\,\frac{dz}{iz} = \frac{1}{i}\oint\limits_{|z|=1} \frac{4z-z^2-1}{5z-2z^2-2}\,\frac{dz}{z}$$

$$= \frac{1}{-2i}\oint\limits_{|z|=1} \frac{4z-z^2-1}{z(z-\frac{1}{2})(z-2)}\,dz \tag{8}$$

Now by Cauchy we can 'shrinkwrap' that last integral about $z = 0$ and $z = 1/2$, giving TWO contributions:

$$\oint\limits_{|z|=\epsilon} \ldots + \oint\limits_{|z-\frac{1}{2}|=\epsilon} \ldots = \frac{1}{-2i}\frac{(-1)}{(-\frac{1}{2})(-2)}2\pi i + \frac{1}{-2i}\frac{(\frac{3}{4})}{(\frac{1}{2})(-\frac{3}{2})}2\pi i = 2\pi. \tag{9}$$

Woaa!... Time for a beer now. Or two. Or two and a pie (OK, stop).