Solutions and Hints for a few complex problems

Exam 4, #2:

1. Matlab chooses $-\pi < \arg(z) \leq \pi$, what value would it give for (with $i^2 = -1$)

1. $|2 + 3i| = \sqrt{13}$

2. $e^{i3\pi/2} = -i \qquad (\ldots \text{but } \arg(e^{i3\pi/2}) = -\pi/2)$

3. $\cos(i) = (e^{-1} + e)/2 = \cosh(1)$

4. $\ln(-i) = -i\pi/2$

5. $\sqrt{1+i} = |1+i|^{1/2}\left(e^{i\pi/4}\right)^{1/2} = 2^{1/4}e^{i\pi/8}$

6. $i^{1+i} = e^{(1+i)\ln i} = e^{(1+i)i\pi/2} = e^{i\pi/2}e^{-\pi/2} = ie^{-\pi/2}$

7. $\arg(i^{1+i}) = \pi/2$

8. $z^{1/3} = |z|^{1/3}e^{i\arg(z)/3}$

9. For which z's, if any, will *Matlab* give $z^{1/3} = i$: NONE

10. If z is any complex number, sketch the domain (or 'range') in the complex w-plane where Matlab will put $w = z^{1/3}$.

    $\arg(w) = \arg(z)/3$ so $-\pi/3 < \arg(w) \leq \pi/3$.

As discussed in earlier solutions and in the complex notes, the complex substitution $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta \Leftrightarrow d\theta = dz/(iz)$ gives $\sin\theta = (z - 1/z)/(2i)$ and

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = 2 \oint_{z=e^{i\theta}} \frac{dz}{bz^2 + 2iaz - b}. \tag{1}$$

Now $bz^2 + 2iaz - b = b\,(z^2 + 2iqz - 1)$, where $q = a/b$ with a, b real.

Next, $z^2 + 2iqz - 1 = 0$ for $z = z_1$ and $z = z_2$ with

$$z_{1,2} = -iq \pm \sqrt{1 - q^2}. \tag{2}$$

If $|q| < 1$ then $1 - q^2 > 0$ and the zeros $z_{1,2}$ are such that $|z_{1,2}| = 1$. They are on the unit circle! (and the integral this is for does not exist).

If $|q| > 1$ then $1 - q^2 < 0$ and the zeros $z_{1,2}$ are pure imaginary. We can write them as:

$$z_{1,2} = i\,(-q \pm \sqrt{q^2 - 1}) \qquad \text{for } \ |q| > 1.$$

One of these is always outside the unit circle, the other one is inside, which one depends on the sign of q. We can sort that out by writing the roots when $|q| > 1$ in the form

$$z_\pm = i\,q\,(-1 \pm \epsilon) \qquad \text{with} \quad \epsilon = \frac{\sqrt{q^2 - 1}}{|q|} > 0 \tag{3}$$

and $0 < \epsilon < 1$ since $|q| > 1$.
Now clearly the root $z_- = i\,q\,(-1 - \epsilon)$ is outside the unit circle, while $z_+ = i\,q\,(-1 + \epsilon)$ MUST be inside since the product of the roots $z_+z_- = -1$. This follows directly from the fact that $(z - z_+)(z - z_-) = z^2 - (z_+ + z_-)z + z_+z_- \equiv z^2 + 2iqz - 1$.

Finally

$$2 \oint_{|z|=1} \frac{dz}{bz^2 + 2iaz - b} = \frac{2}{b} \oint_{|z|=1} \frac{dz}{(z - z_-)(z - z_+)} = \frac{4\pi i}{b} \frac{1}{z_+ - z_-} = \frac{4\pi i}{b} \frac{1}{2i\epsilon q}$$

$$= \frac{2\pi}{a\epsilon} = \frac{2\pi}{a\sqrt{1 - b^2/a^2}}. \tag{4}$$

Only the sign of a matters.