SPOILER WARNING: these pages reveal solutions to some classical problems. You will waste good learning opportunities if you directly read the answers without first attempting the problems yourself. Best strategy is to first try to solve the problem yourself. Give it a good try, reading the problem then saying *'I have no idea how to get started'* because the answer does not immediately pop to your mind is not good enough. You need to try and try again, it's OK to struggle with a problem for 1/2 an hour or an hour or more when you're training your brain. Not only is it OK, it is normal and necessary. There are no shortcuts, if you skip the healthy struggle to solve problems, you are cheating yourself.
There are many ways of solving the same problem. If you get a solution, analyze it carefully to make sure it is correct then compare to the solution provided. Verify your solution again, try to understand the solution provided, see which one is 'better' (*i.e.* shorter, clearer, etc.), try to reproduce the solution from scratch by using different notation, making alternative choices, etc. Do this until the solution is 'obvious' to you. That's what 'understanding' a solution means. That's when you have reached stage 4: *'unconscious competence'*. Don't be stuck at stage 1.

Problem (1.2.6): Show that the medians of a triangle intersect at the same point which is 2/3 of the way down along the median from the vertex.
[A median is a line from a vertex to the middle of the opposite side].

Problem (1.2.7): Given three points A, B, C find a point O such that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$.

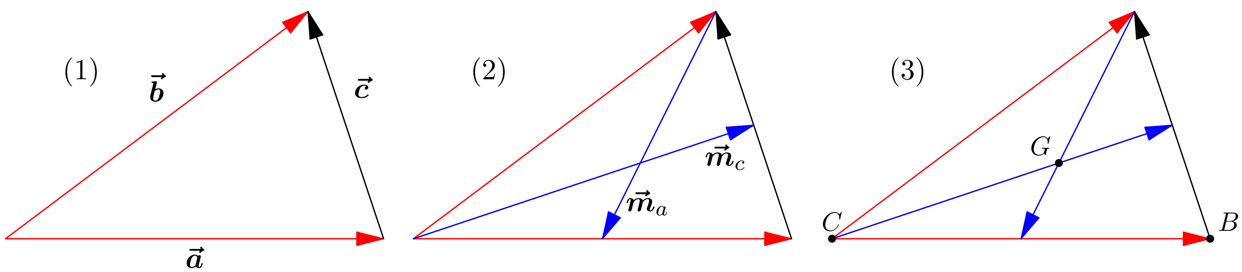
Problem (1.3.7): Show that the normals (or 'heights' or 'altitudes') of a triangle are concurrent. [A 'normal' is the line from a vertex perpendicular to the opposite side, 'concurrent' means that they all pass through the same point.]

Problem (in sect. 1.5): Visualize $(\vec{a} \times \vec{b}) \times \vec{a}$. Describe it geometrically. What is its direction? What is its magnitude? What is its relation to $\vec{b}_\perp$?

TRY THESE YOURSELF BEFORE LOOKING AT THE SOLUTIONS!

Problem (1.2.6): *The medians of a triangle intersect at the same point which is 2/3 of the way down along the median from the vertex.*
SPOILER WARNING: DIGESTED PROBLEM SOLUTION FOLLOWS!

We can solve this using methods of plane Euclidean geometry (*i.e.* similar triangles) but we will use vectors to practice our vector algebra. Start with an arbitrary triangle, don't take a special triangle (rectangular, isosceles, etc.) as this will confuse your thought process, you will assume things that look true on your picture but are not true in general. Don't clobber your picture with so much information that you do not know where to start. You can make several pictures and define objects as needed.



(1) Pick any two sides as *basis* vectors $\vec{a}$ and $\vec{b}$, so the 3rd side is $\vec{c} = -\vec{a} + \vec{b} = \vec{b} - \vec{a}$.

(2) Pick any two medians, for example $\vec{m}_a = -\vec{b} + \frac{1}{2}\vec{a}$ and $\vec{m}_c = \vec{a} + \frac{1}{2}(\vec{b} - \vec{a})$.

(3) Let G be the intersection point of the 2 medians. Then the vector $\overrightarrow{CG}$ can be written in two ways using the two medians:

$$\overrightarrow{CG} = \vec{b} + \alpha\,\vec{m}_a = \beta\,\vec{m}_c,$$

for some unknown real numbers α and β.

(4) Substitute the expressions for the medians $\vec{m}_a$ and $\vec{m}_c$ in terms of the basis vectors $\vec{a}$ and $\vec{b}$ into that vector equation and clean up the algebra:

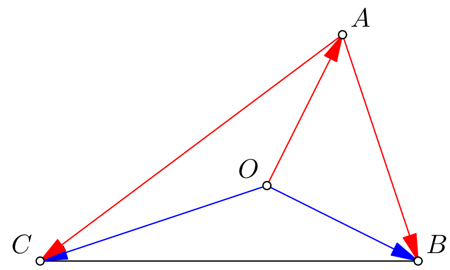
$$\begin{aligned} & \vec{b} + \alpha\vec{m}_a = \beta\vec{m}_c \\ \Rightarrow \quad & \vec{b} + \alpha\left(-\vec{b} + \frac{1}{2}\vec{a}\right) = \beta\left(\vec{a} + \frac{1}{2}(\vec{b} - \vec{a})\right) \\ \Rightarrow \quad & (1 - \alpha - \frac{\beta}{2})\vec{b} = (\beta - \alpha)\frac{1}{2}\vec{a} \end{aligned}$$

Now since $\vec{a}$ and $\vec{b}$ are non-parallel, *i.e.* linearly independent, we must have $(1-\alpha-\beta/2) = \beta-\alpha = 0$ so $\alpha = \beta = 2/3$. This proves that G is 2/3 of the way down from the vertices along the medians.

(5) Since this defines G uniquely, we could take any other two medians $\vec{m}_a$ and $\vec{m}_b$, or $\vec{m}_c$ and $\vec{m}_b$, where $\vec{m}_b = -\vec{a} + \frac{1}{2}\vec{b}$, and the reasoning would go through identically leading to the same intersection point G, 2/3 of the way down along any of the 3 medians.
(5*) Or verify directly that the vector $\overrightarrow{BG}$ slices $\vec{b}$ in half: $\overrightarrow{BG} = \vec{c} + \frac{2}{3}\vec{m}_a = \vec{b} - \vec{a} - \frac{2}{3}\vec{b} + \frac{1}{3}\vec{a} = \frac{1}{3}\vec{b} - \frac{2}{3}\vec{a}$ and this is indeed 2/3 of the median $\vec{m}_b$ from B since $\vec{m}_b = -\vec{a} + \frac{1}{2}\vec{b}$ so $\frac{2}{3}\vec{m}_b = \frac{2}{3}\left(-\vec{a} + \frac{1}{2}\vec{b}\right) = \frac{1}{3}\vec{b} - \frac{2}{3}\vec{a} = \overrightarrow{BG}$.

Problem (1.2.7): *Given three points A, B, C find a point O such that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$.*
SPOILER WARNING: DIGESTED PROBLEM SOLUTION FOLLOWS!

This is a variant of the medians problem. A, B, C define a triangle. A little thought leads to the conclusion that O must be inside that triangle although that is not needed for the solution.



The vector equation $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0$ appears to have three unknown vectors since we do not know point O, however that is the only unknown point since A, B and C are given. So use the hint to rewrite two of the unknown vectors in terms of the third, for instance: $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$, $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$, where $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are known. The vector equation is then easily solved

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0 \quad \Rightarrow \; 3\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{AC} = 0$$

$$\Rightarrow \; -\overrightarrow{OA} = \overrightarrow{AO} = \frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{AC}\right).$$

Now this vector $\overrightarrow{AO}$ is $2/3$ of the median from vertex A to side (B, C) since that median vector is $\frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$.
If that last statement is not obvious, here are the gory details: that median is $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$, but $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{AC} - \overrightarrow{AB}$, so the median is

$$\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AB} + \frac{1}{2}\left(\overrightarrow{AC} - \overrightarrow{AB}\right) = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right).$$

So this point O is the same as the point we called G in the solution of problem 1.2.6.

Problem (1.3.7): *The normals of a triangle are concurrent.*
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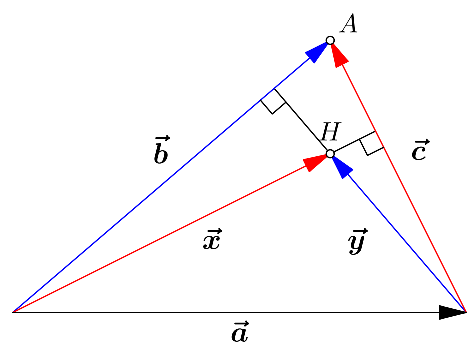
We could use an approach similar to the medians problem, pick any two normals, find their intersection then verify that the vector from the third vertex to that intersection point is perpendicular to the third side. But that involves more algebra than necessary since we do not need to find the intersection point explicitly.



(1) Pick any two normals and let $\vec{x}$ and $\vec{y}$ be the vectors from their respective vertex to the intersection point H.

(2) For the choices in the figure above, $\vec{x}$ is perpendicular to $\vec{c}$ and $\vec{y}$ is perpendicular to $\vec{b}$ so

$$\vec{x} \cdot \vec{c} = 0, \qquad \vec{y} \cdot \vec{b} = 0$$

(3) We would like to show that $\vec{a} \cdot \overrightarrow{AH} = 0$.

(4) We need to rewrite $\vec{a}$ and $\overrightarrow{AH}$ in terms of the vectors that define the known normals $\vec{b}$, $\vec{y}$, and $\vec{c}$, $\vec{x}$ in order to use the orthogonality of these pairs, for instance

$$\vec{a} = \vec{x} - \vec{y}, \qquad \overrightarrow{AH} = -\vec{b} + \vec{x} = -\vec{c} + \vec{y},$$

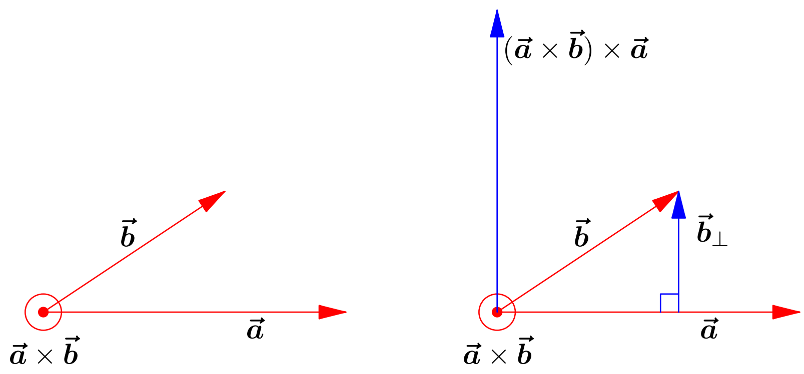
then

$$\begin{aligned}
\vec{a} \cdot \overrightarrow{AH} =& (\vec{x} - \vec{y}) \cdot \overrightarrow{AH} = \vec{x} \cdot \overrightarrow{AH} - \vec{y} \cdot \overrightarrow{AH} \\
=& \vec{x} \cdot (\vec{y} - \vec{c}) - \vec{y} \cdot (\vec{x} - \vec{b}) \\
=& \vec{x} \cdot \vec{y} - \vec{y} \cdot \vec{x} \qquad\qquad (*) \\
=& 0.
\end{aligned}$$

(*) Hey! where did $-\vec{x} \cdot \vec{c}$ and $\vec{y} \cdot \vec{b}$ go?!

Problem (in sect. 1.5): *Visualize* $(\vec{a}\times\vec{b})\times\vec{a}$. *Describe it geometrically. What is its direction? What is its magnitude? What is its relation to* $\vec{b}_\perp$*?*

SPOILER WARNING: DIGESTED PROBLEM SOLUTION FOLLOWS!



(1) The cross-product $(\vec{a}\times\vec{b})$ is perpendicular to $\vec{a}$ and $\vec{b}$.

(2) Now consider $(\vec{a}\times\vec{b})\times\vec{a}$, that's perpendicular to $(\vec{a}\times\vec{b})$ so it is back in the $\vec{a}$, $\vec{b}$ plane. It is also perpendicular to $\vec{a}$ so it is in the direction of $\vec{b}_\perp$.

(3) Now for the magnitude: $\vec{a}\times\vec{b} = \vec{a}\times\vec{b}_\perp$ and

$$|\vec{a}\times\vec{b}| = |\vec{a}\times\vec{b}_\perp| = |\vec{a}|\,|\vec{b}_\perp|$$

since by definition $\vec{a}$ and $\vec{b}_\perp$ are perpendicular. Likewise $(\vec{a}\times\vec{b})$ and $\vec{a}$ are perpendicular so

$$|(\vec{a}\times\vec{b})\times\vec{a}| = |\vec{a}\times\vec{b}|\,|\vec{a}| = |\vec{a}|\,|\vec{b}_\perp|\,|\vec{a}| = (\vec{a}\cdot\vec{a})\,|\vec{b}_\perp|$$

(4) Since the direction of $(\vec{a}\times\vec{b})\times\vec{a}$ is the same as that of $\vec{b}_\perp$ we have

$$\boxed{(\vec{a}\times\vec{b})\times\vec{a} = (\vec{a}\cdot\vec{a})\,\vec{b}_\perp}$$

Of course we can divide by $\vec{a}\cdot\vec{a}$ to get a cool formula for $\vec{b}_\perp$

$$\vec{b}_\perp = \frac{(\vec{a}\times\vec{b})\times\vec{a}}{\vec{a}\cdot\vec{a}}$$

It is pretty similar to the formula for the parallel component

$$\vec{b}_\parallel = \frac{(\vec{a}\cdot\vec{b})}{\vec{a}\cdot\vec{a}}\,\vec{a}.$$

These formula are relatively easy to reconstruct once you understand the dot and cross products. Study the figures above until they become second nature to you.