NO BOOK, NO NOTES, NO CALCULATOR. READ CAREFULLY!!!
Use clean and precise notation. Clearly show the steps in your reasoning.

Some *potentially* but not *necessarily* useful formula:

$$\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j, \tag{1}$$

$$\epsilon_{ijk} = (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k, \tag{2}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{3}$$

$$\boldsymbol{\nabla} \cdot (\varphi \boldsymbol{a}) = (\boldsymbol{\nabla}\varphi) \cdot \boldsymbol{a} + \varphi\,(\boldsymbol{\nabla} \cdot \boldsymbol{a})\,, \tag{4}$$

$$\boldsymbol{\nabla} \times (\varphi \boldsymbol{a}) = (\boldsymbol{\nabla}\varphi) \times \boldsymbol{a} + \varphi\,(\boldsymbol{\nabla} \times \boldsymbol{a})\,, \tag{5}$$

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \tag{6}$$

provided $\mathcal{C}$ encloses z and $f(z)$ is analytic inside $\mathcal{C}$.

DIC= Discussed In Class; SE= Suggested Exercise.

1. *Calculate, justifying briefly (remember that a repeated index means a summation over all values of that index)*

(a) $\epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$ (b) $\epsilon_{ijk}\epsilon_{ijk} = 6$

(c) $\boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = 1/2\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{a}) - (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{a}$

If a_{ij} are the components of a 3-by-3 real matrix,

(d) what is the geometrical meaning of $\epsilon_{ijk}\, a_{i1}a_{j2}a_{k3}$? It is the *signed volume* (*i.e.* the determinant) of the parallelepiped whose sides are given by the three columns of the matrix.

(e) Calculate $\epsilon_{ijk}\, a_{i1}\, a_{j1}\, a_{k3}$. The first 2 columns are repeated, so that is ZERO. Also, using indicial manipulations: $\epsilon_{ijk}\, a_{i1}\, a_{j1}\, a_{k3} = \epsilon_{jik}\, a_{j1}\, a_{i1}\, a_{k3} = \epsilon_{jik}\, a_{i1}\, a_{j1}\, a_{k3} = -\epsilon_{ijk}\, a_{i1}\, a_{j1}\, a_{k3}$.

Where the heck does this come from? (a) and (b) are SE 2.9.3(c) and (d) which provide the answers and were DIC; they can be derived directly or by calculation from formula (3) recalling from 2.9.3(a) that $\delta_{ii} = 3$; (c) was DIC and is the identity given in SE 1.8.13. It can be done using the above formula (3) used in class several times. (d) and (e) follow directly from the formula for mixed products and determinants (formula 33 in vector notes), see also book page 145 ("Levi-Civita Symbol").

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2. *In classical mechanics, the motion of a particle of mass m is governed by Newton's law $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{F}$ is the net force acting on the particle and $\boldsymbol{a}(t) = d\boldsymbol{v}/dt$ is the particle's acceleration with $\boldsymbol{r}(t)$ its position vector and $\boldsymbol{v}(t) = d\boldsymbol{r}/dt$ its velocity vector.*

(a) Find the most general $\boldsymbol{r}(t)$ *when* $\boldsymbol{F} \equiv 0$*. What type of motion is this?*

(b) If $\boldsymbol{v}(t) = \boldsymbol{\omega} \times \boldsymbol{r}(t)$ *where* $\boldsymbol{\omega}$ *is a constant vector, show that* $\boldsymbol{a}$ *is orthogonal to* $\boldsymbol{v}$*. What type of motion is this?*

(c) Pick an orthonormal basis $\hat{\boldsymbol{x}}$*,* $\hat{\boldsymbol{y}}$*,* $\hat{\boldsymbol{z}}$*, such that* $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ *and find an explicit parametrization for* $\boldsymbol{r}(t)$*.*

A particle of electric charge q *moving in a magnetic field* $\boldsymbol{B}$ *experiences the Lorentz force* $\boldsymbol{F} = q(\boldsymbol{v} \times \boldsymbol{B})$*. Write Newton's law for such a particle, then for constant* $\boldsymbol{B}$

(d) what is the motion in the direction parallel to the magnetic field $\boldsymbol{B}$*?*

(e) what is the motion in planes perpendicular to $\boldsymbol{B}$*?*

(a) $\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\boldsymbol{v}_0$ where $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are constant vectors. This is free or linear motion (particle moves on a line).
(b) $\boldsymbol{a} = d\boldsymbol{v}/dt = \boldsymbol{\omega} \times d\boldsymbol{r}/dt = \boldsymbol{\omega} \times \boldsymbol{v}$ because $d\boldsymbol{\omega}/dt = 0$. Hence $\boldsymbol{a} \cdot \boldsymbol{v} = 0$ because $\boldsymbol{\omega} \times \boldsymbol{v}$ is orthogonal to both $\boldsymbol{\omega}$ and $\boldsymbol{v}$ by definition of cross product. This is circular motion corresponding to right-hand rotation about the direction of $\boldsymbol{\omega}$ at rate $\|\boldsymbol{\omega}\|$. The distance to the axis remains constant. The velocity in the direction of $\boldsymbol{\omega}$ is zero.
(c) With $\boldsymbol{\omega} = \omega\hat{\boldsymbol{z}}$ then $\boldsymbol{v} = d\boldsymbol{r}/dt = \omega\hat{\boldsymbol{z}} \times \boldsymbol{r}$. Let $\boldsymbol{r}(t) = \hat{\boldsymbol{x}}x(t) + \hat{\boldsymbol{y}}y(t) + \hat{\boldsymbol{z}}z(t)$ where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are *fixed*, orthonormal basis vectors, then from (b) we know that $z(t) = \textit{constant}$ and $x^2(t) + y^2(t) = R^2 = \textit{constant}$ so $x(t)$ and $y(t)$ are the coordinates of a point on a circle. Now the rate of rotation must be ω so

$$x(t) = \cos\omega t, \quad y(t) = \sin\omega t, \quad z(t) = z_0.$$

We can also rederive everything:

$$\frac{d\boldsymbol{r}}{dt} = \hat{\boldsymbol{x}}\dot{x} + \hat{\boldsymbol{y}}\dot{y} + \hat{\boldsymbol{z}}\dot{z} = \omega\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z) = \omega x\hat{\boldsymbol{y}} - \omega y\hat{\boldsymbol{x}},$$

therefore

$$\dot{x} = -\omega y, \quad \dot{y} = \omega x, \quad \dot{z} = 0, \Rightarrow \ddot{x} + \omega^2 x = 0, \quad z(t) = z_0$$

$x(t)$ and $y(t)$ satisfy the harmonic oscillator equation and $x(t) = \cos\omega t$, $y = \sin\omega t$.
(d) $\boldsymbol{a} = d\boldsymbol{v}/dt = (q/m)\boldsymbol{v} \times \boldsymbol{B}$, so $\boldsymbol{a}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{B}$. For constant $\boldsymbol{B}$, this means that the *velocity component* in the direction of $\boldsymbol{B}$ is constant.
(e) For constant $\boldsymbol{B}$ we have circular motion (by (b) and (c) above) about the diredction $\boldsymbol{B}$ at angular velocity $\omega = (q/m)\|\boldsymbol{B}\|$. (d) and (e) together mean that the charged particle follows a *helical path.*

Where the heck does this come from? see Section 1.10 "Motion of a particle" in vector notes. (b) and (c) are also SE 1.7.1.

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3. *(a) Carefully deduce and explain (using clear diagrams and formula and a few words), the general formula for the right-hand rotation* $R(\boldsymbol{a}) \equiv \boldsymbol{R} \cdot \boldsymbol{a}$ *of the arbitrary vector* $\boldsymbol{a}$ *by an angle* θ *about and arbitrary unit vector* $\boldsymbol{n}$*. The formula with no clear explanation will earn little credit.*

(b) Express the components R_{ij} of the rotation tensor $\boldsymbol{R}$ in terms of θ and the components of the direction vector $\boldsymbol{n}$ (for an arbitrary orthonormal basis). [Using the Kronecker and Levi-Civita symbols δ_{ij} and ϵ_{ijk}].

This must have been done at least three times in class and many more times during regular and irregular office hours. See notes on tensors and exercise 8 there.

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4. *Consider the force field $\boldsymbol{F} = r^n \boldsymbol{r}$, where $\boldsymbol{r}$ is the position vector, $r = \|\boldsymbol{r}\|$ and n is any integer ($n = 0, \pm 1, \pm 2, \cdots$). Find (for all such n!)*

(a) $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, *(b)* $\boldsymbol{\nabla} \times \boldsymbol{F}$, *(c)* φ *such that* $\boldsymbol{F} = -\boldsymbol{\nabla}\varphi$

(d) The flux $\oint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ *where S is the surface of a sphere of radius R centered at the origin.*

(e) The flux $\oint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ *through ANY closed surface S when $n = 0$.*

(f) The work $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ *where $\mathcal{C}$ is the circle of radius R about the origin.*

Where the heck does this come from? This is SE 1.13.1 and is also closely related to problems 2 and 3 on Exam 2. See again solved Examples 1.7.1, 1.7.2, 1.8.1, 1.9.1. Formula (4) and (5) given above are quite useful.

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5. *(a) Find all the roots of $z^3 = -8$.*

(b) Find all the roots of $e^z = -e$, where $e = \exp(1) = 2.71828....$

Now this should have been an easy 10 out of 70 points, but roughly 75% of the class bombed them!

(a) $z = 2e^{i(2k+1)\pi/3} = -2, 1 \pm i\sqrt{3}$. (b) $z = 1 + i(2k+1)\pi$.

Where the heck does this come from? you were reminded in class about the roots of unity, *i.e.* all the solutions of $z^n = 1$. This comes up in several problems where we need to find poles of a function to do contour integration. Also came up when we discussed branch points. See also SE 6.1.5. (b) is rather fundamental and is basically SE 6.1.13.

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6. *Show that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$. Derive the explicit form of that polynomial for any positive integer n.*

Where the heck does this come from? Did you check out the solutions to exam 2? A (very) few of you did. This is basically problem 5 on exam 2. It's also Example 6.1.1 and exercise 6.1.6 in the book where the solution is outlined.

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7. *Calculate (1)* **OR** *(2) for $a > 1$, justifying carefully [***both** *for extra credit!]*

$$(1) \quad \int_{-\infty}^{\infty} \frac{\cos ax - \cos x}{x^2}\, dx \qquad OR \qquad (2) \quad \int_0^{\pi} \frac{d\theta}{a + \cos\theta}.$$

Where the heck does this come from?

(1) This is SE 7.2.11, the solution is very similar to that for the very classical $\int_{-\infty}^{\infty}(\sin x)/x\,dx$ done in the book (example 7.2.3) and in class. Consider $\int_C e^{iaz}/z^2 dz$ with the same contour as for $(\sin x)/x$ (figure 7.8 in book). There is a 2nd order pole at $z = 0$. You do not even need Jordan's Lemma to justify that the integral over C_2 goes to zero. The final answer (given in the book) is $\pi(1-a)$.

(2) This is a slight rewrite of example 7.2.1. which calculates explicitly $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\epsilon\cos\theta}$, and a special case of SE 7.2.7 $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, which we solved explictly in class.
Of course

$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$

(see also SE 7.2.8, 7.2.10 for that tricky trigonometric trick), so you consider an integral over the entire unit circle, not just half of it.