1. *If δ_{ij} is the well-known Kronecker symbol and ϵ_{ijk} is the permutation (or Levi-Civita) symbol well-known to Math 321 students, the two symbols are related by the identity (using summation convention)*

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \tag{1}$$

Let φ, v_k, $\boldsymbol{a}$, $\boldsymbol{b}$ be arbitrary functions of the position vector $\boldsymbol{r}$, calculate, justifying briefly:

(a) $\epsilon_{132} = -1$, (b) $\epsilon_{122} = 0$,
(c) This *scalar* is equal to minus itself, hence it is zero:

$$\epsilon_{ijk}\frac{\partial^2 v_k}{\partial x_i \partial x_j} = \epsilon_{ijk}\frac{\partial^2 v_k}{\partial x_j \partial x_i} = \epsilon_{jik}\frac{\partial^2 v_k}{\partial x_i \partial x_j} = -\epsilon_{ijk}\frac{\partial^2 v_k}{\partial x_i \partial x_j}.$$

(d) $\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{b}) \equiv \epsilon_{ijk}\partial_j(\epsilon_{klm}a_l b_m) = \epsilon_{ijk}\epsilon_{klm}\partial_j(a_l b_m) = \epsilon_{kij}\epsilon_{klm}\partial_j(a_l b_m)$

$= [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}]\partial_j(a_l b_m) = \partial_m(a_i b_m) - \partial_l(a_l b_i) = a_i(\partial_m b_m) + (b_m\partial_m)a_i - (a_l\partial_l)b_i - b_i(\partial_l a_l)$

$$\equiv \boldsymbol{a}(\boldsymbol{\nabla}\cdot\boldsymbol{b}) + (\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{b} - \boldsymbol{b}(\boldsymbol{\nabla}\cdot\boldsymbol{a}).$$

(e) show that $\boldsymbol{\nabla}\cdot(\varphi\boldsymbol{a}) = (\boldsymbol{\nabla}\varphi)\cdot\boldsymbol{a} + \varphi\,(\boldsymbol{\nabla}\cdot\boldsymbol{a})$ using indicial notation.

$$\boldsymbol{\nabla}\cdot(\varphi\boldsymbol{a}) \equiv \partial_i(\varphi a_i) = (\partial_i\varphi)a_i + \varphi(\partial_i a_i) \equiv (\boldsymbol{\nabla}\varphi)\cdot\boldsymbol{a} + \varphi\,(\boldsymbol{\nabla}\cdot\boldsymbol{a}).$$

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2. *Let $\boldsymbol{r}$ be the position vector in 3D space and r its magnitude. The gradient vector operator is denoted $\boldsymbol{\nabla}$. If $f(r)$ is an arbitrary twice differentiable function of r, calculate the following expressions showing or stating the key steps of your reasoning. Your writing must clearly distinguish between scalars, vectors and tensors!!*

These are solved explicitly in the book, see Examples 1.6.1, 1.7.1, 1.8.1, 1.9.1 and were solved in class as well.
(a) $\boldsymbol{\nabla}f(r) = (df/dr)\boldsymbol{r}/r$, (b) $\boldsymbol{\nabla}\cdot\boldsymbol{r} = 3$, (c) $\boldsymbol{\nabla}\times\boldsymbol{\nabla}f(r) = 0$, (d) $\boldsymbol{\nabla}\cdot(\boldsymbol{r}/r^3) = 0$ except at $r=0$, (e) $\boldsymbol{\nabla}\times(\boldsymbol{r}/r^3) = 0$, except at $r = 0$.

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3. *Calculate, justifying carefully,*
(a) $\oint_S \boldsymbol{n}\cdot\boldsymbol{\nabla}\varphi\,dS$, *where S is the closed surface surrounding a volume V where $\boldsymbol{\nabla}^2\varphi = 5$, and $\boldsymbol{n}$ is the unit outward normal to S.*
By the divergence theorem $\oint_S \boldsymbol{n}\cdot\boldsymbol{\nabla}\varphi\,dS = \int_V \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\varphi)\,dV = 5V$. The operator $\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ appears in many suggested exercises.

(f) $\oint_S (\boldsymbol{r}/r^3)\cdot d\boldsymbol{S}$ *where S is the surface of the sphere of radius R centered at $x_c = R/2$, $y_c = z_c = 0$ and $d\boldsymbol{S} \equiv \boldsymbol{n}dS$ is the surface element with $\boldsymbol{n}$ the unit outward normal.*
This surface encloses the origin where the $\boldsymbol{r}/r^3$ diverges and its divergence is not defined. So we cannot directly use the divergence/Gauss theorem. But we can use it in a domain that excludes the origin, as done in class and in section 1.14. The result is "Gauss's Law" and the intergal is 4π. Physically, this is the flux due to a unit "point charge" at $r = 0$.

(g) $\oint_{\mathcal{C}} (\boldsymbol{r}/r^3) \cdot d\boldsymbol{r}$ *where $\mathcal{C}$ is the circle of radius 4 centered at $x = 2$, $y = 1$.*
By Stokes theorem and 2 (e), $\oint_{\mathcal{C}} (\boldsymbol{r}/r^3) \cdot d\boldsymbol{r} = \int_A \left(\boldsymbol{\nabla} \times \frac{\boldsymbol{r}}{r^3} \right) \cdot \boldsymbol{n} dA = 0.$

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4. *Find the Taylor series of $1/(1+x^2)$ about $x = 0$ and specify its domain of convergence (i.e. the domain of x where the series converges). Justify.*
This is Math 221/222 material. We reviewed the key ideas in class.

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

This series converges if $|x| < 1$, and now that you've studied functions of one complex variable, you understand more deeply why this restriction occurs.

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5. *Show that*

$$\sin(n+1)\theta = \sin\theta \, P_n(\cos\theta), \tag{2}$$

where $P_n(x)$ is a polynomial of degree n (integer) in x and find $P_n(x)$ explicitely. ($P_n(x)$ is the Chebyshev polynomial of the second kind.).
This is suggested problem 6.1.6. See Example 6.1.1 "De Moivre's formula" for the basic idea and exercise 6.1.6 for the answer.

$$e^{in\theta} = \left(e^{i\theta}\right)^n \Leftrightarrow \cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n.$$

The binomial formula gives that

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}, \quad \text{where } C_n^k \equiv \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}.$$

The C_n^k are called the binomial coefficients. Using that formula, we get

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} C_n^k \, i^k (\sin\theta)^k (\cos\theta)^{n-k}.$$

So $\sin n\theta$ is equal to the imaginary part of that sum. To extract the imaginary part we need only consider $k = 2l+1$ odd because $i^{2l+1} = (-1)^l i$ and any even power of i is real. Let $n = 2M$ or $n = 2M+1$ (depending on whether n is even or odd), then

$$\sin n\theta = \sin\theta \sum_{l=0}^{M} C_n^{2l+1} (-1)^l (\sin\theta)^{2l} (\cos\theta)^{n-2l-1}.$$

The sum can be rewritten as a polynomial in $\cos\theta$ because $\sin^2\theta = 1 - \cos^2\theta$, so

$$\sin n\theta = \sin\theta \sum_{l=0}^{M} C_n^{2l+1} (-1)^l (1-\cos^2\theta)^l (\cos\theta)^{n-2l-1}.$$