2.2+ Gradient extras

Geometric definition of gradient: Given a (sufficiently nice) scalar field $f(\vec{r})$, *e.g.* temperature as a function of position, its gradient $\vec{\nabla} f$ at point $\vec{r}$ is *a vector pointing in the direction of greatest increase of f*. The magnitude of $\vec{\nabla} f$ is *the rate of change of f with distance in that direction*. It follows that the gradient at a point is perpendicular to the isosurface of f that passes through that point.

Fundamental examples:

(1) If $f = f(r)$ where $r = |\vec{r}|$, the scalar field depends only on distance to the origin, then f is constant if r is constant, so the isosurfaces are spheres, $\vec{\nabla} f(r)$ is in the radial direction $\hat{\boldsymbol{r}}$ and its magnitude is the rate of change in the radial direction df/dr so

$$\vec{\nabla} f(r) = \frac{df}{dr}\hat{\boldsymbol{r}}.$$

(2) If $f = f(|\vec{r} - \vec{r}_c|)$ so the scalar field depends only on distance to point $\vec{r}_c$, then f is constant if $|\vec{r} - \vec{r}_c|$ is constant, so the isosurfaces are spheres centered at $\vec{r}_c$ and, letting $\vec{r} - \vec{r}_c = s\,\hat{\boldsymbol{s}}$ with $s = |\vec{r} - \vec{r}_c|$

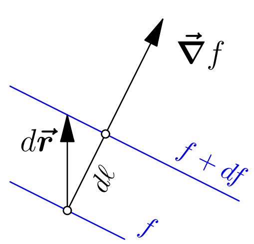
$$\vec{\nabla} f(s) = \frac{df}{ds}\hat{\boldsymbol{s}} = \frac{df}{ds}\,\frac{\vec{r} - \vec{r}_c}{|\vec{r} - \vec{r}_c|}.$$

(3) If $f = f(\vec{r} \cdot \vec{c})$ where $\vec{c}$ is a fixed vector, then f is constant if $\vec{r} \cdot \vec{c}$ is constant, so the isosurfaces are planes perpendicular to $\vec{c}$. Let $s = \vec{r} \cdot \vec{c} = |\vec{c}|\ell$ where ℓ is distance in the $\vec{c}$ direction so

$$\vec{\nabla} f(\vec{r} \cdot \vec{c}) = \frac{df}{d\ell}\,\frac{\vec{c}}{|\vec{c}|} = \frac{df}{ds}\,\vec{c}.$$

In particular if $\vec{c} = \hat{\boldsymbol{x}}$ then $\vec{r} \cdot \hat{\boldsymbol{x}} = x$ and $\vec{\nabla} f(x) = \hat{\boldsymbol{x}}\,df/dx$.

It also follows from the geometric definition that the differential change df in the value of f when $\vec{r}$ changes by an arbitrary differential $d\vec{r}$ is



$$\boxed{df = d\vec{r} \cdot \vec{\nabla} f} \tag{1}$$

That is because, locally, the isosurfaces are planes perpendicular to the gradient, so a displacement $d\vec{r}$ leads to a change in f in proportion to the component of the displacement in the direction of the gradient $d\ell = d\vec{r}\cdot(\vec{\nabla} f)/|\vec{\nabla} f|$. The change in f is $df = d\ell\,|\vec{\nabla} f| = d\vec{r}\cdot\vec{\nabla} f$.

That general relationship (1) between df and $d\vec{r}$ allows us to obtain the expression for the $\vec{\nabla} f$ in various sets of coordinates, including non-cartesian coordinates.

Gradient in cartesian coordinates

The position vector reads $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Picking $d\vec{\boldsymbol{r}} = \hat{\boldsymbol{x}}\,dx$, $d\vec{\boldsymbol{r}} = \hat{\boldsymbol{y}}\,dy$ and $d\vec{\boldsymbol{r}} = \hat{\boldsymbol{z}}\,dz$ in (1), respectively, gives

$$\frac{\partial f}{\partial x} = \frac{\partial \vec{\boldsymbol{r}}}{\partial x} \cdot \vec{\boldsymbol{\nabla}} f = \hat{\boldsymbol{x}} \cdot \vec{\boldsymbol{\nabla}} f, \tag{2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial \vec{\boldsymbol{r}}}{\partial y} \cdot \vec{\boldsymbol{\nabla}} f = \hat{\boldsymbol{y}} \cdot \vec{\boldsymbol{\nabla}} f, \tag{3}$$

$$\frac{\partial f}{\partial z} = \frac{\partial \vec{\boldsymbol{r}}}{\partial z} \cdot \vec{\boldsymbol{\nabla}} f = \hat{\boldsymbol{z}} \cdot \vec{\boldsymbol{\nabla}} f. \tag{4}$$

These three relationships imply

$$\vec{\boldsymbol{\nabla}} f = \hat{\boldsymbol{x}}\frac{\partial f}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial f}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}. \tag{5}$$

Gradient in spherical coordinates

Here $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, so

$$\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}} = r\left(\hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta\right), \tag{6}$$

where r is the distance to the origin, θ is the polar angle (co-latitude) and ϕ is the azimuthal angle (longitude).
In earlier sections on spherical coordinates and volume parametrizations, we discussed/derived

$$\frac{\partial \vec{\boldsymbol{r}}}{\partial r} = \hat{\boldsymbol{r}}, \quad \frac{\partial \vec{\boldsymbol{r}}}{\partial \theta} = r\,\hat{\boldsymbol{\theta}}, \quad \frac{\partial \vec{\boldsymbol{r}}}{\partial \phi} = r\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{7}$$

These results can be obtained via the hybrid cartesian/spherical expression (6) for $\vec{\boldsymbol{r}}$ or directly from the geometry and understanding of partial derivatives (see (14, 15) below). Then, from (1) and (7),

$$\frac{\partial f}{\partial r} = \frac{\partial \vec{\boldsymbol{r}}}{\partial r} \cdot \vec{\boldsymbol{\nabla}} f = \hat{\boldsymbol{r}} \cdot \vec{\boldsymbol{\nabla}} f, \tag{8}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial \vec{\boldsymbol{r}}}{\partial \theta} \cdot \vec{\boldsymbol{\nabla}} f = r\hat{\boldsymbol{\theta}} \cdot \vec{\boldsymbol{\nabla}} f, \tag{9}$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial \vec{\boldsymbol{r}}}{\partial \phi} \cdot \vec{\boldsymbol{\nabla}} f = r\sin\theta\hat{\boldsymbol{\phi}} \cdot \vec{\boldsymbol{\nabla}} f. \tag{10}$$

Since $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are orthonormal, these expressions imply that

$$\vec{\boldsymbol{\nabla}} f = \hat{\boldsymbol{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}. \tag{11}$$

Note that the gradient in spherical coordinates is *not* $\vec{\boldsymbol{\nabla}} f = \hat{\boldsymbol{r}}\partial f/\partial r + \hat{\boldsymbol{\theta}}\,\partial f/\partial \theta + \hat{\boldsymbol{\phi}}\,\partial f/\partial \phi$! That expression is not even dimensionally correct.

To derive the spherical coordinates expression for other operators such as divergence $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$, curl $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$ and Laplacian $\nabla^2 = \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}}$, one needs to know the rate of change of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ with the coordinates (r, θ, ϕ). These vectors change with (r, θ, ϕ) unlike the cartesian direction vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ which are the same at every point.
Since a partial with respect to r means the rate of change in a fixed radial direction (θ, ϕ fixed), it should be clear geometrically that $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ do not change with r

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial r} = \frac{\partial \hat{\boldsymbol{\theta}}}{\partial r} = \frac{\partial \hat{\boldsymbol{\phi}}}{\partial r} = 0. \tag{12}$$

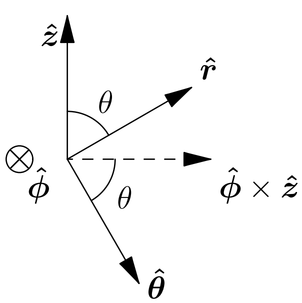
To deduce the rates of change with respect to θ and ϕ, we could start from the hybrid expression (6) for $\vec{\boldsymbol{r}}(r, \theta, \phi)$ and crank it out, but a much faster, geometric approach is to use our knowledge of rotation: if a vector $\vec{\boldsymbol{v}}$ rotates about $\vec{\boldsymbol{\omega}}$ then its governing equation is $d\vec{\boldsymbol{v}}/dt = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}$ where t is time. In differential form, this is $d\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} dt \times \vec{\boldsymbol{v}}$ or

$$d\vec{\boldsymbol{v}} = d\alpha\, \hat{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}} \tag{13}$$

where $d\alpha = |\vec{\boldsymbol{\omega}}|\, dt$ is the differential angle of rotation during the time interval dt.
The $\partial/\partial\theta$ derivatives correspond to 'infinitesimal' rotation in meridional planes since r and ϕ are fixed. This is rotation by $d\theta$ about $\hat{\boldsymbol{\phi}}$ hence from (13), we obtain (14)

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \theta} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}}, \quad \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{r}}, \quad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = 0. \tag{14}$$

Likewise $\partial/\partial\phi$ corresponds to 'infinitesimal' rotation about $\hat{\boldsymbol{z}}$ by angle $d\phi$, hence from (13),



$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \phi} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \sin\theta\, \hat{\boldsymbol{\phi}}, \tag{15}$$

$$\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{\theta}} = \cos\theta\, \hat{\boldsymbol{\phi}}, \tag{16}$$

$$\frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{\phi}} = -\sin\theta\, \hat{\boldsymbol{r}} - \cos\theta\, \hat{\boldsymbol{\theta}}. \tag{17}$$

Exercises:

1. What is $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$ in spherical coordinates where $\vec{\boldsymbol{v}} = \hat{\boldsymbol{r}}\, u + \hat{\boldsymbol{\theta}}\, v + \hat{\boldsymbol{\phi}}\, w$?
2. Derive the gradient in cylindrical coordinates and the derivatives of the cylindrical direction vectors.