In response to **Jacob Schmid**'s question about radius of convergence of a Taylor series, I discussed Taylor series for the *Lorentzian* function $f(x) = \frac{1}{1+x^2}$ with x real. This is a good old real f of a real x. It qualitatively looks like a Gaussian function e^{-x^2} but it decays much more slowly as $|x| \to \infty$.

As your learned in Math 221 and 222, the power series expansion of a function $f(z)$ about $z = a$, that is the expansion of $f(z)$ in powers of $(z - a)$, is the Taylor Series of $f(z)$ about a:

$$\begin{aligned} f(z) &= c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (z-a)^n \\ &= f(a) + f'(a)(z-a) + \frac{1}{2} f''(a)(z-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n, \end{aligned} \tag{1}$$

where $f^{(n)}(a)$ is the nth derivative of $f(z)$ at a and $n! = n(n-1)\cdots 1$ is the factorial of n, with $0! = 1$ by convenient definition. The equality between $f(z)$ and its Taylor series is only valid if the series converges.

For the real Lorentzian function $f(x)$ we find

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^8 + x^{10} - \cdots = \sum_{n=0}^{\infty} (-x^2)^n \tag{2}$$

This is the Taylor series of $1/(1 + x^2)$ about $x = 0$. We computed it without calculating any derivative! We did that using the geometric series (yes, that one again! it really is a useful little series). From the geometric series we know that it converges if $|-x^2| < 1$, that is if $|x| < 1$. But why?! the function $1/(1+x^2)$ is very nice, there is nothing nasty occuring at $x = \pm 1$! why does the series stop converging there?! See the figure (1) for Taylor approximations about $x = 0$ in action.

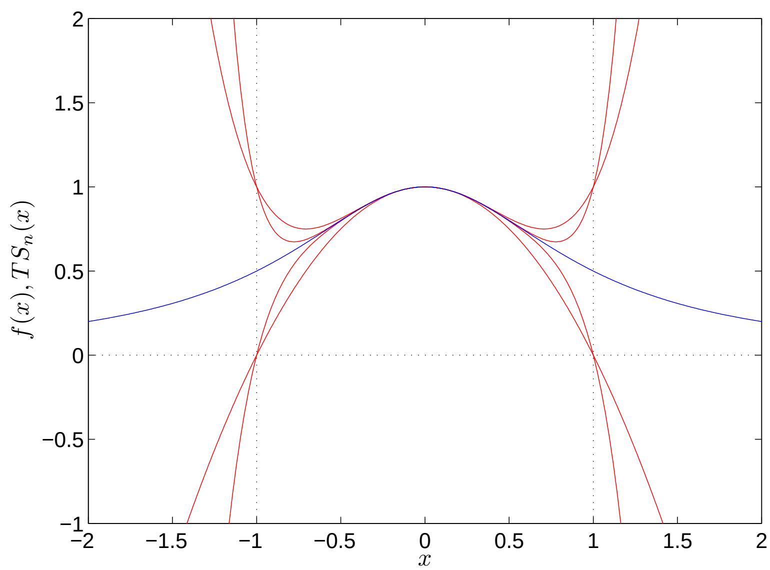

Figure 1: The Lorentzian $f(x) = 1/(1+x^2)$ in blue and its Taylor approximations $1-x^2$, $1-x^2+x^4$, $1-x^2+x^4-x^6$, $1-x^2+x^4-x^6+x^8$ in red. These simple polynomial approximations do an increasingly better job of approximating $f(x)$ near $x = 0$ but they blow up beyond $x = \pm 1$.

Now if we expand $f(x)$ in powers of $(x-1)$ we get

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4 + \frac{1}{8}(x-1)^5 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{\sin\left((n+1)3\pi/4\right)}{(\sqrt{2})^{n+1}}\,(x-1)^n \\ &= \left[\frac{1}{2} - \frac{x-1}{2} + \frac{(x-1)^2}{4}\right]\left[1 - \frac{(x-1)^4}{4} + \frac{(x-1)^8}{16} - \cdots\right] \\ &= \left[\frac{1}{2} - \frac{x-1}{2} + \frac{(x-1)^2}{4}\right]\sum_{m=0}^{\infty}\left(\frac{-(x-1)^4}{4}\right)^m.\end{aligned} \tag{3}$$

This Taylor approximation of the same $f(x) = 1/(1+x^2)$ but now in powers of $(x-1)$, not x, is illustrated in figure 2. By the good old ratio test or geometric series directly, we know the series converges for $|x-1| < \sqrt{2}$. But why $\sqrt{2}$ now?!

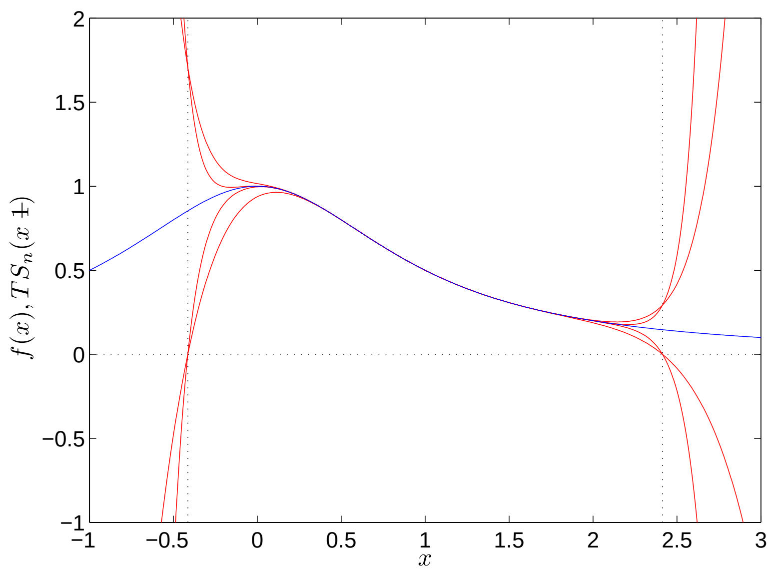

Figure 2: The Lorentzian $f(x) = 1/(1+x^2)$ in blue and its Taylor approximations (3) in red for $n = 0$ to $n = 6$, 10, 14 and 18. These simple polynomial approximations do an increasingly better job of approximating $f(x)$ near $x = 1$ but they blow up beyond $x = 1 \pm \sqrt{2}$. Why $\sqrt{2}$?!

Ask **Jacob Schmid**, he should know!