FW Math 321 **SOLUTIONS: Exam 1, Thur. Oct. 8, 2009**

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1. *Index notation and Einstein's summation convention:*

(a) Define δ_{ij}. What are i, j?

$\boxed{\delta_{ij} = 1 \text{ if } i = j,\ \delta_{ij} = 0 \text{ if } i \neq j, \text{ where } i = 1,2 \text{ or } 3 \text{ and } j = 1,2, \text{ or } 3.}$

(b) Define ϵ_{ijk}. What are i, j and k?

$\epsilon_{ijk} = 1$ if (i,j,k) is an even permutation of (1,2,3),$= -1$ for odd permutation, =0 otherwise.
Examples: $\epsilon_{231} = 1$, $\epsilon_{321} = -1$, $\epsilon_{331} = 0$.

(c) What is δ_{kk} =? Explain/show key steps. $\boxed{\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.}$

(d) What is $\epsilon_{ijk} - \epsilon_{kji}$ =? Explain.

$\epsilon_{kji} = -\epsilon_{ijk}$ since (k,j,i) is an odd permutation of (i,j,k). So $\epsilon_{ijk} - \epsilon_{kji} = 2\epsilon_{ijk}$.

(e) Complete the following formula

$$\begin{aligned}\epsilon_{ijk}\,\epsilon_{lmn} =&\delta\quad\delta\quad\delta\quad+\delta\quad\delta\quad\delta\quad+\delta\quad\delta\quad\delta\\ &-\delta\quad\delta\quad\delta\quad-\delta\quad\delta\quad\delta\quad-\delta\quad\delta\quad\delta\end{aligned}$$

(formula (45) reconstructed in class)

$$\begin{aligned}\epsilon_{ijk}\,\epsilon_{lmn} =&\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km}\\ &- \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km}\end{aligned}$$

(note: cyclic with +, acyclic with −).

(f) Use that formula to calculate $\epsilon_{mjk}\epsilon_{lmn}$. Be concise but clearly show key steps.

(problem 1.6 #4, done in class)

$$\begin{aligned}\epsilon_{mjk}\,\epsilon_{lmn} =&\delta_{ml}\delta_{jm}\delta_{kn} + \delta_{mm}\delta_{jn}\delta_{kl} + \delta_{mn}\delta_{jl}\delta_{km}\\ &- \delta_{mm}\delta_{jl}\delta_{kn} - \delta_{mn}\delta_{jm}\delta_{kl} - \delta_{ml}\delta_{jn}\delta_{km}\\ =&\delta_{jl}\delta_{kn} + 3\delta_{jn}\delta_{kl} + \delta_{jl}\delta_{kn}\\ &- 3\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl} - \delta_{jn}\delta_{kl}\\ =&\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}\end{aligned}$$

(which you should be able to write down almost directly).

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2. *Index notation and Einstein's summation convention:*

(i) What does $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ for any $i = 1,2,3$ and $j = 1,2,3$, say about $\vec{e}_1, \vec{e}_2, \vec{e}_3$?

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ are ORTHONORMAL, *i.e.* mutually orthogonal unit vectors.

(ii) If *furthermore*, $(\vec{e}_i \times \vec{e}_j) \cdot \vec{e}_k = \epsilon_{ijk}$, what does this mean about $\vec{e}_1, \vec{e}_2, \vec{e}_3$?

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, in that order, are RIGHT HANDED.

(iii) If $\vec{\boldsymbol{u}} = u_i \vec{\boldsymbol{e}}_i$ and $\vec{\boldsymbol{v}} = v_i \vec{\boldsymbol{e}}_i$, write $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ and $(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}})_i$ in terms of the u_i and v_i's.

$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = u_i \vec{\boldsymbol{e}}_i \times v_j \vec{\boldsymbol{e}}_j = \epsilon_{ijk} u_i v_j \vec{\boldsymbol{e}}_k$ (since (ii) implies that $\vec{\boldsymbol{e}}_i \times \vec{\boldsymbol{e}}_j = \epsilon_{ijk} \vec{\boldsymbol{e}}_k$).

$(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}})_i = \vec{\boldsymbol{e}}_i \cdot (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) = \epsilon_{lmn} u_l v_m (\vec{\boldsymbol{e}}_n \cdot \vec{\boldsymbol{e}}_i) = \epsilon_{lmn} u_l v_m \delta_{in} = \epsilon_{lmi} u_l v_m = \epsilon_{ijk} u_j v_k$

(iv) If furthermore $\vec{\boldsymbol{w}} = w_i \vec{\boldsymbol{e}}_i$, write $(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \times \vec{\boldsymbol{w}}$ and $(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}}$ in terms of the u_i, v_i and w_i's. Simplify as much as possible using (i) and (ii) above.

$(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \times \vec{\boldsymbol{w}} = \epsilon_{ijk} u_i v_j \vec{\boldsymbol{e}}_k \times w_l \vec{\boldsymbol{e}}_l = \epsilon_{ijk} u_i v_j w_l \epsilon_{klm} \vec{\boldsymbol{e}}_m = \epsilon_{ijk} \epsilon_{klm} u_i v_j w_l \vec{\boldsymbol{e}}_m$ (all dummies. The indices I mean).

$(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}} = (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}})_i w_i = \epsilon_{ijk} u_j v_k w_i = \epsilon_{jki} u_j v_k w_i = \epsilon_{ijk} u_i v_j w_k$ (any one of the latter 3 OK).

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3. (i) If $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ are arbitrary vectors in 3D space, with length a and b respectively, prove that the vector $a \vec{\boldsymbol{b}} + b \vec{\boldsymbol{a}}$ makes equal angles with $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$.

(ii) Three points A, B and C have cartesian coordinates (A_1, A_2, A_3), (B_1, B_2, B_3) and (C_1, C_2, C_3), respectively. *Derive* cartesian equations for the line in the A, B, C plane that bisects the angle $\widehat{BAC}$.

(i) how could we figure out angle between two vectors? Dot product maybe?

Let $\vec{\boldsymbol{v}} = a\vec{\boldsymbol{b}} + b\vec{\boldsymbol{a}}$ then $\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{a}} = v \cos\alpha$ and $\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{b}} = v \cos\beta$, so $\alpha = \beta$ if $\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{a}} = \vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{b}}$. Let's check: $\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{a}} = a\vec{\boldsymbol{b}} \cdot \hat{\boldsymbol{a}} + ab = ab(\hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{a}} + 1)$ and $\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{b}} = ab + b\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} = ab(1 + \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}})$, so yes $\alpha = \beta$.

(ii) The line is the set of point P such that $\overrightarrow{AP} = t\vec{\boldsymbol{v}}$ with $\vec{\boldsymbol{v}} = a\vec{\boldsymbol{b}} + b\vec{\boldsymbol{a}}$ as in (i) and $\vec{\boldsymbol{a}} = \overrightarrow{AB}$, $\vec{\boldsymbol{b}} = \overrightarrow{AC}$. Since $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{r}}_A$, the line is $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_A + t\vec{\boldsymbol{v}}$.

Now in cartesian coords
- ▷ $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \equiv x_i \vec{\boldsymbol{e}}_i$ (as Einstein would write it)
- ▷ $\vec{\boldsymbol{a}} = \overrightarrow{AB} = (B_1 - A_1)\hat{\boldsymbol{x}} + (B_2 - A_2)\hat{\boldsymbol{y}} + (B_3 - A_3)\hat{\boldsymbol{z}} \equiv (B_i - A_i)\vec{\boldsymbol{e}}_i$, or $a_i = B_i - A_i$.
- ▷ $\vec{\boldsymbol{b}} = \overrightarrow{AC} = (C_1 - A_1)\hat{\boldsymbol{x}} + (C_2 - A_2)\hat{\boldsymbol{y}} + (C_3 - A_3)\hat{\boldsymbol{z}} \equiv (C_i - A_i)\vec{\boldsymbol{e}}_i$, or $b_i = C_i - A_i$.
- ▷ $a = |\vec{\boldsymbol{a}}| = \sqrt{a_i a_i} = \sqrt{a_1^2 + a_2^2 + a_3^2}$, likewise $b = \sqrt{b_i b_i}$

So the line $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_A + t\vec{\boldsymbol{v}}$ written in cartesian coordinates is $x_i = A_i + t(ab_i + ba_i)$ or in all details

$$x = A_1 + t(ab_1 + ba_1), \quad y = A_2 + t(ab_2 + ba_2), \quad z = A_3 + t(ab_3 + ba_3),$$

with a, b and a_i, b_i as defined above.

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4. When the Whooping Cranes migrate from Necedah, WI to St. Marks National Wildlife Refuge in Florida, what is the minimum distance they must travel?
Derive a formula to compute that distance. (1) Specify what reasonable geographic data you must know. (2) Specify how to derive the distance from that data. Your final answer should be a complete algorithm to calculate the distance from the data.

OLD NEWS: This is problem 1.3 #8, HWK 3, 9/10/2009.

BIG NEWS: Earth is not flat! It's pretty close to a sphere of radius R ($\approx 6378km$). Positions on earth are given in terms of *longitude*, φ say, and *latitude*, λ say.

Necedah's position (point A) is (φ_1, λ_1), St. Marks (point B) is (φ_2, λ_2).

The shortest distance D is the arc of great circle on the sphere: $D = R\theta$, where θ is the angle between $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$, the radius vectors of points A and B with the origin at the center of the earth. Thus $\vec{\boldsymbol{r}}_1 \cdot \vec{\boldsymbol{r}}_2 = R^2 \cos\theta$. To compute this dot product, we switch to cartesian coordinates (Math 321 Lecture 1: cartesian, cylindrical and spherical coordinates)

$x_1 = \rho_1 \cos\varphi_1$, $y_1 = \rho_1 \sin\varphi_1$, $z_1 = R\sin\lambda_1$, with $\rho_1 = R\cos\lambda_1$.
$x_2 = \rho_2 \cos\varphi_2$, $y_2 = \rho_2 \sin\varphi_2$, $z_2 = R\sin\lambda_2$, with $\rho_2 = R\cos\lambda_2$.

So the distance D is about $R\theta = R \arccos\left(\dfrac{\vec{\boldsymbol{r}}_1 \cdot \vec{\boldsymbol{r}}_2}{R^2}\right) = R\arccos\left(\dfrac{x_1x_2 + y_1y_2 + z_1z_2}{R^2}\right)$.

If you really want to know $(\varphi_1, \lambda_1) = (-90.074, 44.026))$, $(\varphi_2, \lambda_2) = (-84.206, 30.161)$ so $\theta \approx 0.2552$ radians and $D \approx 1,628$ km $\approx 1,011$ miles. Sounds reasonable. Google maps says 1,140 miles walking.

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5. A particle with radius vector $\vec{\boldsymbol{r}}(t)$ moves according to the equation $\vec{\boldsymbol{v}} \doteq d\vec{\boldsymbol{r}}/dt = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$, where $\vec{\boldsymbol{\omega}}$ is a known, constant vector.
(i) Derive an expression for $\vec{\boldsymbol{a}}(t) = d\vec{\boldsymbol{v}}/dt$, *write your expression in the simplest form with the clearest geometric interpretation.*

(ii) Make *two* clean sketches that *clearly* illustrate the relationships between $\vec{\boldsymbol{\omega}}$, $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$.

(see problem 1.5 #5, done in class + HWK 8, #6)

$a = d\vec{\boldsymbol{v}}/dt = d/dt(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) = \vec{\boldsymbol{\omega}} \times d\vec{\boldsymbol{r}}/dt = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) = \vec{\boldsymbol{\omega}}(\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{r}}) - \vec{\boldsymbol{r}}(\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{\omega}}) = -\omega^2\, \vec{\boldsymbol{r}}_\perp$

You're on your own for the sketches, I've done enough such plots in class with SIDE and TOP views.