FW Math 321 Exam 2, Thur. Nov. 19, 2009

NAME and Major: BRIEF SOLUTIONS

1. An ellipse in the (x, y) plane is centered at (x_c, y_c), its major axis is a and points in a direction that makes an angle α counterclockwise from the x-axis, its minor axis is b.

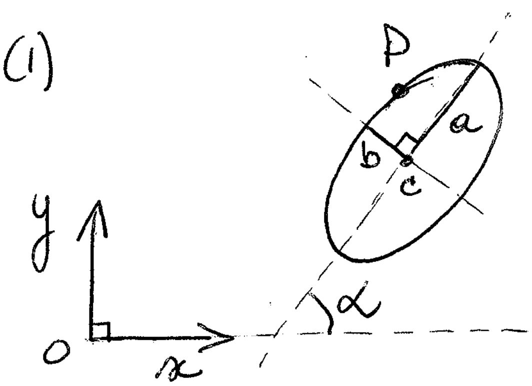
(1) SKETCH, showing all data

(2) What is the vector parametric equation $\vec{r} = \vec{r}(t)$ for the ellipse? Make sure to properly define/indicate on your sketch everything that is needed.

(3) What are the explicit cartesian parametric equations $x = x(t)$, $y = y(t)$ for the ellipse? specify $x(t)$ and $y(t)$, what is the range of t?

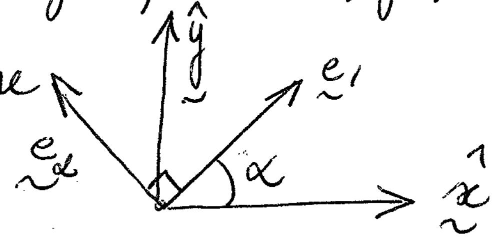
(4) What is the t-integral that one needs to compute to find the length of the ellipse?

(1)



$P \equiv (x, y)$, $C \equiv (x_c, y_c)$

Define



so $\begin{cases} \underset{\sim}{e}_1 = \cos\alpha\, \hat{\underset{\sim}{x}} + \sin\alpha\, \hat{\underset{\sim}{y}} \\ \underset{\sim}{e}_\alpha = -\sin\alpha\, \hat{\underset{\sim}{x}} + \cos\alpha\, \hat{\underset{\sim}{y}} \end{cases}$

(2) $\underset{\sim}{r} = \overrightarrow{OP} = \overrightarrow{OC} + \underset{\sim}{e}_1\, a\cos t + \underset{\sim}{e}_\alpha\, b\sin t$, $t = 0 \to 2\pi$

(3) $x = \underset{\sim}{r} \cdot \hat{\underset{\sim}{x}} = x_c + a\cos\alpha\cos t - b\sin\alpha\sin t$

$y = \underset{\sim}{r} \cdot \hat{\underset{\sim}{y}} = y_c + a\sin\alpha\cos t + b\cos\alpha\sin t$

(4) $L = \int_{\mathcal{C}} |d\underset{\sim}{r}| = \int_0^{2\pi} \left|\frac{d\underset{\sim}{r}}{dt}\right| dt = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

or from (2) directly: $|d\underset{\sim}{r}| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$

$\Rightarrow L = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\, dt$

(x_c, y_c, α drop out of L!)

2. (a) Complete the following expression

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

(b) use it to DERIVE a vector identity for $\vec{\nabla} \times (\vec{v} \times \vec{w})$ that has no cross products

$$\underset{\sim}{\nabla} \times (\underset{\sim}{v} \times \underset{\sim}{w}) \triangleq \underset{\sim}{u} \quad \text{(vector field)}$$

$$\Rightarrow \quad u_i = \varepsilon_{ijk}\, \partial_j (\varepsilon_{klm} v_l w_m)$$

$$= \varepsilon_{ijk}\, \varepsilon_{klm}\, \partial_j (v_l w_m)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(w_m \partial_j v_l + v_l \partial_j w_m)$$

$$= w_j \partial_j v_i - w_i \partial_j v_j + v_i \partial_j w_j - v_j \partial_j w_i$$

$$\equiv (\underset{\sim}{w} \cdot \underset{\sim}{\nabla})\underset{\sim}{v} - \underset{\sim}{w}(\underset{\sim}{\nabla} \cdot \underset{\sim}{v}) + \underset{\sim}{v}(\underset{\sim}{\nabla} \cdot \underset{\sim}{w}) - (\underset{\sim}{v} \cdot \underset{\sim}{\nabla})\underset{\sim}{w}$$

3.

$$\vec{r} = r\hat{\boldsymbol{r}} = \rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

is the position vector (a.k.a. radius vector) in spherical (r, θ, φ), cylindrical (ρ, φ, z) and cartesian (x, y, z) coordinates, respectively.

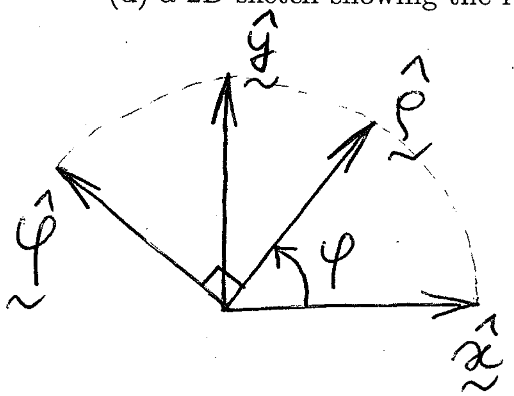
Write ρ and r in terms of cartesian coordinates (x, y, z):

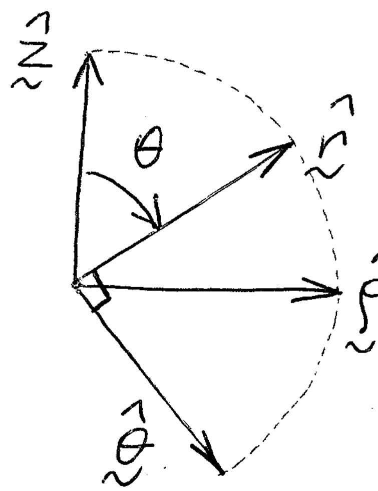
(a) $\rho = \sqrt{x^2+y^2}$ (b) $r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$

Draw side by side:

(c) a 2D sketch showing the relationship between $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$. Indicate φ and $\hat{\boldsymbol{\varphi}}$.

(d) a 2D sketch showing the relationship between $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{z}}$. Indicate θ and $\hat{\boldsymbol{\theta}}$.





Write $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\varphi}}$ in terms of ρ, φ, z and the cartesian basis vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$.

(e) $\hat{\boldsymbol{\rho}} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$ (f) $\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

Calculate and express in cylindrical coordinates ρ, φ, z with cylindrical basis vectors $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{z}}$

(g) $\frac{\partial\hat{\boldsymbol{\rho}}}{\partial\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y} = \hat{\varphi}$ (h) $\frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\varphi} = -\cos\varphi\,\hat{x} - \sin\varphi\,\hat{y} = -\hat{\rho}$

4. $\vec{r} = r\hat{r} = \rho\hat{\rho} + z\hat{z} = x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector in spherical, cylindrical and cartesian coordinates, respectively, $\vec{\nabla}$ is the del operator such that $\vec{\nabla} f$ is the gradient of f, Santa Claus told you that $\vec{\nabla} \times (\hat{\varphi}/\rho) = 0$, except on the polar axis (of course, Ho! Ho! Ho!), and α is a scalar constant.

Calculate (JUSTIFY/SHOW STEPS!)

$$\text{(a) } \vec{\nabla} r^{\alpha}, \qquad \text{(b) } \vec{\nabla} \rho^{\alpha}, \qquad \text{(c) } \vec{\nabla} \cdot \hat{r}, \qquad \text{(d) } \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^3}\right), \qquad \text{(e) } \vec{\nabla} \times \left(\frac{\hat{\varphi}}{\rho^3}\right)$$

write the results in whatever coordinates and basis are most appropriate or convenient.

(a) $\underset{\sim}{\nabla} r^{\alpha} = \hat{\underset{\sim}{r}} \dfrac{d}{dr} r^{\alpha} = \alpha r^{\alpha-1} \hat{\underset{\sim}{r}}$ spherical function isosurfaces $\equiv$ spheres

(b) $\underset{\sim}{\nabla} \rho^{\alpha} = \hat{\underset{\sim}{\rho}} \dfrac{d}{d\rho} \rho^{\alpha} = \alpha \rho^{\alpha-1} \hat{\underset{\sim}{\rho}}$ cylindrical function isosurfaces $\equiv$ cylinders

(c) $\underset{\sim}{\nabla} \cdot \hat{\underset{\sim}{r}} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{r}/r) = \dfrac{1}{r} \underset{\sim}{\nabla} \cdot \underset{\sim}{r} + \underset{\sim}{r} \cdot \underset{\sim}{\nabla} r^{-1}$

$$= \frac{3}{r} - \frac{\underset{\sim}{r} \cdot \hat{\underset{\sim}{r}}}{r^2} = \frac{3}{r} - \frac{1}{r} = \boxed{\frac{2}{r}}$$

(d) $\underset{\sim}{\nabla} \cdot (\hat{\underset{\sim}{r}}/r^3) = \dfrac{1}{r^4} \underset{\sim}{\nabla} \cdot \underset{\sim}{r} + \underset{\sim}{r} \cdot \underset{\sim}{\nabla} r^{-4}$

$$= \frac{3}{r^4} - \frac{4}{r^5} \underset{\sim}{r} \cdot \hat{\underset{\sim}{r}} = \boxed{-1/r^4}$$

(e) $\underset{\sim}{\nabla} \times (\hat{\underset{\sim}{\varphi}}/\rho^3) = \underset{\sim}{\nabla} \times (\underset{\sim}{B}/\rho^2) = \dfrac{1}{\rho^2} \underbrace{\underset{\sim}{\nabla} \times \underset{\sim}{B}}_{=0} + \underbrace{(\underset{\sim}{\nabla} \rho^{-2})}_{\frac{-2}{\rho^3}\hat{\underset{\sim}{\rho}}} \times \underset{\sim}{B}$

(with $\underset{\sim}{B} = \dfrac{1}{\rho} \hat{\underset{\sim}{\varphi}}$)

$$= -\frac{2}{\rho^4} \hat{\underset{\sim}{\rho}} \times \hat{\underset{\sim}{\varphi}} = \boxed{-\frac{2}{\rho^4} \hat{\underset{\sim}{z}}}$$

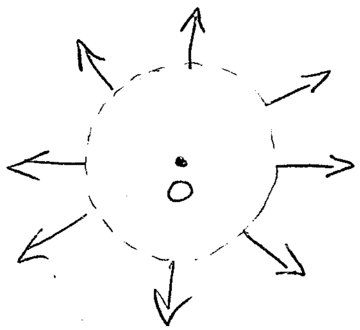
5. Calculate the flux $F = \oint_S \vec{v} \cdot \overrightarrow{dS}$ where $\vec{v} = \hat{r}/r^3$ as in problem 2(d) and $\overrightarrow{dS}$ is pointing outward,

(a) when S is the sphere of radius R centered at $r = 0$. JUSTIFY/SHOW STEPS

(b) Is F positive, negative or zero when S is the *sphere* of radius R centered at the point $x = 2R$, $y = 0$, $z = 0$? JUSTIFY

(c) Specify *completely* what Fundamental Theorem you are using *if any*.

(a)



$\underset{\sim}{dS} = \underset{\sim}{\hat{r}}\, dS$ for this sphere

$\Rightarrow \underset{\sim}{v} \cdot \underset{\sim}{dS} = \frac{1}{R^3}\, dS$

$$F = \frac{4\pi R^2}{R^3} = \boxed{4\pi/R}$$

(b)

Now $\underset{\sim}{v}$ and $\underset{\sim}{\hat{n}}$ are NOT parallel.
But singular point O is outside sphere

by (c) $F = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dV = \int_V \left(-\frac{1}{r^4}\right) dV < 0$

(c) <u>Divergence theorem:</u>

$$\int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{v}\, dV = \oint_S \underset{\sim}{v} \cdot \underset{\sim}{\hat{n}}\, dS$$

where S is surface enclosing volume V and $\underset{\sim}{\hat{n}}$ is unit outward normal.

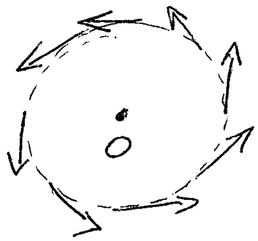
6. Calculate the circulation $\Gamma = \oint_C \vec{v} \cdot d\vec{r}$ where $\vec{v} = \hat{\varphi}/\rho^3$ as in problem 2(e)

(a) when C is the circle of radius R in the x, y plane, centered at $x = 0$, $y = 0$, $z = 0$ and oriented counterclockwise in (x, y)? JUSTIFY/SHOW STEPS

(b) is Γ positive, negative or zero when C is a *circle* of radius R centered at $x = 2R$, $y = 0$, $z = 0$ and oriented counterclockwise in (x, y)? JUSTIFY

(c) Specify *completely* what Fundamental Theorem you are using *if any.*

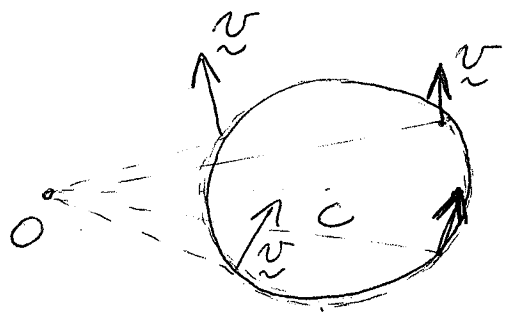
(a)



on $\mathcal{C}$: $\underset{\sim}{v} \parallel d\underset{\sim}{r} \Rightarrow \underset{\sim}{v} \cdot d\underset{\sim}{r} = \frac{1}{R^3} |d\underset{\sim}{r}|$

$$\Rightarrow \Gamma = \frac{1}{R^3} \oint_{\mathcal{C}} |d\underset{\sim}{r}| = \frac{2\pi R}{R^3} = \boxed{\frac{2\pi}{R^2}}$$

(b)



Now $\underset{\sim}{v}$ is NOT $\parallel$ $d\underset{\sim}{r}$

but from $\mathcal{C}$

$$\Gamma = \int_A \underset{\sim}{\nabla} \times \underset{\sim}{v} \cdot \hat{\underset{\sim}{z}} \, dA$$

$$= \int_A -\frac{2}{\rho^4} \, dA < 0$$

(c) Stokes Theorem:

$$\int_S (\underset{\sim}{\nabla} \times \underset{\sim}{v}) \cdot \hat{\underset{\sim}{n}} \, dS = \oint_{\mathcal{C}} \underset{\sim}{v} \cdot d\underset{\sim}{r}$$

where $\mathcal{C}$ is the boundary curve of the surface S and the orientation of $\mathcal{C}$ matches $\hat{\underset{\sim}{n}}$ for S by the right hand rule