The *tensor product* is another way to multiply vectors, in addition to the dot and cross products. The tensor product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted $\boldsymbol{a} \otimes \boldsymbol{b}$ in mathematics but simply $\boldsymbol{ab}$ with no special product symbol in mechanics. The result of the tensor product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is not a scalar, like the dot product, nor a (pseudo)-vector like the cross-product. It is a new object called a *tensor of second order* $\boldsymbol{ab}$ that is defined indirectly through the following dot products between the tensor $\boldsymbol{ab}$ and *any* vector $\boldsymbol{v}$:

$$(\boldsymbol{ab}) \cdot \boldsymbol{v} \equiv \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{v}), \qquad \boldsymbol{v} \cdot (\boldsymbol{ab}) \equiv (\boldsymbol{v} \cdot \boldsymbol{a})\boldsymbol{b}, \qquad \forall \boldsymbol{v}. \tag{1}$$

The right hand sides of these equations are readily understood. These definitions clearly imply that $\boldsymbol{ab} \neq \boldsymbol{ba}$, the tensor product does not commute. However,

$$\boldsymbol{v} \cdot (\boldsymbol{ab}) = (\boldsymbol{ba}) \cdot \boldsymbol{v} \equiv (\boldsymbol{ab})^T \cdot \boldsymbol{v}, \qquad \forall \boldsymbol{v}. \tag{2}$$

The product $\boldsymbol{ba}$ is the *transpose* of $\boldsymbol{ab}$, denoted with a 'T' superscript: $(\boldsymbol{ab})^T \equiv \boldsymbol{ba}$.
We also deduce the following distributive properties, from (1):

$$(\boldsymbol{ab}) \cdot (\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha(\boldsymbol{ab}) \cdot \boldsymbol{u} + \beta(\boldsymbol{ab}) \cdot \boldsymbol{v}, \tag{3}$$

which hold for any scalars α, β and vectors $\boldsymbol{u}, \boldsymbol{v}$.

A general *tensor of 2nd order* can be defined similarly as an object $\boldsymbol{T}$ such that $\boldsymbol{T} \cdot \boldsymbol{v}$ is a vector and

$$\boldsymbol{T} \cdot (\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha \boldsymbol{T} \cdot \boldsymbol{u} + \beta \boldsymbol{T} \cdot \boldsymbol{v}, \qquad \forall \alpha, \beta, \boldsymbol{u}, \boldsymbol{v}. \tag{4}$$

Thus $\boldsymbol{T} \cdot \boldsymbol{v}$ is a *linear transformation* of $\boldsymbol{v}$ and $\boldsymbol{T}$ is a *linear operator.*
The linearity property (4) allows us to figure out the effect of $\boldsymbol{T}$ on any vector $\boldsymbol{v}$ once we know its effect on three non co-planar vectors. Therefore $\boldsymbol{T}$ is fully determined once we know, or specify, the three vectors $\boldsymbol{t}_j \equiv \boldsymbol{T} \cdot \boldsymbol{e}_j$, $j = 1, 2, 3$. Each of these vectors $\boldsymbol{t}_j$ has three components, T_{ij} such that

$$\boldsymbol{T} \cdot \boldsymbol{e}_j \equiv \boldsymbol{t}_j = T_{ij} \boldsymbol{e}_i, \tag{5}$$

(sum over i) and

$$T_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{t}_j = \boldsymbol{e}_i \cdot (\boldsymbol{T} \cdot \boldsymbol{e}_j) \tag{6}$$

if the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is orthogonal. These T_{ij}'s are the 9 components of the tensor $\boldsymbol{T}$ with respect to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. They fully define $\boldsymbol{T}$. Indeed using the summation convention and the linearity property (4), for any vector $\boldsymbol{v} = v_j \boldsymbol{e}_j$ (sum over j) we get

$$\boldsymbol{T} \cdot \boldsymbol{v} = \boldsymbol{T} \cdot (v_j \boldsymbol{e}_j) = v_j \boldsymbol{T} \cdot \boldsymbol{e}_j = T_{ij} v_j \boldsymbol{e}_i. \tag{7}$$

(the last term is a double sum over i and j!).
The sum and product of tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ are defined by $(\boldsymbol{T} + \boldsymbol{S}) \cdot \boldsymbol{v} \equiv \boldsymbol{T} \cdot \boldsymbol{v} + \boldsymbol{S} \cdot \boldsymbol{v}$ and $(\boldsymbol{T} \cdot \boldsymbol{S}) \cdot \boldsymbol{v} \equiv \boldsymbol{T} \cdot (\boldsymbol{S} \cdot \boldsymbol{v})$. Note that the dot product of two second order tensors is a second order tensor and that product does not commute $\boldsymbol{T} \cdot \boldsymbol{S} \neq \boldsymbol{S} \cdot \boldsymbol{T}$.
The tensor $\boldsymbol{T}$ can be expressed as the following linear combination of the 9 tensor products $\boldsymbol{e}_i \boldsymbol{e}_j$, $i, j = 1, 2, 3$ between the basis vectors:

$$\boldsymbol{T} = T_{ij}\, \boldsymbol{e}_i \boldsymbol{e}_j. \tag{8}$$

This is checked easily as follows:

$$(T_{ij} \boldsymbol{e}_i \boldsymbol{e}_j) \cdot \boldsymbol{e}_k = T_{ij} \boldsymbol{e}_i (\boldsymbol{e}_j \cdot \boldsymbol{e}_k) = T_{ij} \boldsymbol{e}_i \delta_{jk} = T_{ik} \boldsymbol{e}_i \equiv \boldsymbol{T} \cdot \boldsymbol{e}_k,$$

from (5). This check also shows that the tensor expansion formula (8) and (6), hold only for an orthogonal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ although the tensor $\boldsymbol{T}$ itself, and the number of its components, do not depend on the properties of any particular basis.

The 9 components T_{ij} form the 3-by-3 matrix of components of the tensor $\boldsymbol{T}$ with respect to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. Likewise the tensor product $\boldsymbol{ab}$ can be represented in terms of that basis as

$$\boldsymbol{ab} = a_i b_j \; \boldsymbol{e}_i \boldsymbol{e}_j. \tag{9}$$

(double sum over i, j) where $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_j \boldsymbol{e}_j$. More explicitly, the components of the tensor $\boldsymbol{ab}$ in the orthogonal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, consist of a 3-by-3 matrix obtained through the "row-by-column" product of column (a_1, a_2, a_3) with the row (b_1, b_2, b_3)

$$\boldsymbol{ab} \equiv \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1 \; b_2 \; b_3) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix}. \tag{10}$$

In particular, $\boldsymbol{e}_1\boldsymbol{e}_1$, $\boldsymbol{e}_1\boldsymbol{e}_2$, and $\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3$ for instance, have components

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

in the orthogonal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, respectively. This last tensor is the *identity tensor*

$$\boldsymbol{I} = \delta_{ij} \; \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{e}_i \boldsymbol{e}_i \equiv \boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3. \tag{12}$$

This is the only tensor such that $\boldsymbol{I} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{I} = \boldsymbol{v}$, $\forall \boldsymbol{v}$.

The transpose of tensor $\boldsymbol{T}$, denoted $\boldsymbol{T}^T$, is defined through the double dot product with any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\boldsymbol{u} \cdot (\boldsymbol{T} \cdot \boldsymbol{v}) \equiv \boldsymbol{v} \cdot (\boldsymbol{T}^T \cdot \boldsymbol{u}), \qquad \forall \boldsymbol{u}, \boldsymbol{v}. \tag{13}$$

The transpose of $\boldsymbol{T} = T_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$ can be written explicitly as

$$\boldsymbol{T}^T = T_{ij} \; \boldsymbol{e}_j \boldsymbol{e}_i = T_{ji} \; \boldsymbol{e}_i \boldsymbol{e}_j. \tag{14}$$

In either case, if T_{ij} are the components of $\boldsymbol{T}$, then the components of $\boldsymbol{T}^T$ are T_{ji}. A tensor is *symmetric* if it equals its transpose, *i.e.* if $\boldsymbol{T} = \boldsymbol{T}^T$ (*e.g.* $\boldsymbol{I}$ is symmetric). It is antisymmetric if it is equal to minus its transpose, *i.e.* if $\boldsymbol{T} = -\boldsymbol{T}^T$. Any tensor can be decomposed into a symmetric part and an antisymmetric part

$$\boldsymbol{T} = \frac{1}{2}(\boldsymbol{T} + \boldsymbol{T}^T) + \frac{1}{2}(\boldsymbol{T} - \boldsymbol{T}^T).$$

One antisymmetric tensor of particular interest is the antisymmetric part of the tensor product $\boldsymbol{ab}$:

$$\boldsymbol{ab} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba}) + \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba}). \tag{15}$$

It is left to the reader to verify that

$$(\boldsymbol{ab} - \boldsymbol{ba}) \cdot \boldsymbol{c} = \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}), \qquad \forall \boldsymbol{c}. \tag{16}$$

This relationship leads to a generalization of the cross-product $\boldsymbol{a} \times \boldsymbol{b}$ in terms of the antisymmetric part of the tensor product, $\boldsymbol{ab} - \boldsymbol{ba}$, for dimensions higher than 3.

Transformation theory of tensors

In Cartesian coordinates, the basis vectors are orthogonal and constant. We can easily "hide" them and focus on the components as the latter determine everything following well-known formulas for dot product, cross-product, etc. The tensor product of the vectors with Cartesian components a_i and b_j gives the tensor $a_i b_j$, $\forall i, j = 1, 2, 3$. Vectors have 3 components denoted using one free index, such as a_i for $i = 1, 2, 3$. These components transform like the coordinates under orthogonal transformation of the axes (*i.e.* rotations and reflections). If $x'_i = Q_{ki} x_k$ (sum over k) corresponds to a change of orthogonal basis, where $Q_{ki} = \hat{\boldsymbol{x}}_k \cdot \hat{\boldsymbol{x}}'_i$ is the matrix of direction cosines, then the components of the vector $\boldsymbol{a} = a_i \hat{\boldsymbol{x}}_i = a'_i \hat{\boldsymbol{x}}'_i$ are related as $a'_i = Q_{ki} a_k$. Likewise $b'_j = Q_{lj} b_l$ for the vector $\boldsymbol{b} = b_j \hat{\boldsymbol{x}}_j = b'_j \hat{\boldsymbol{x}}'_j$ and the components of the tensor product in the new basis are $a'_i b'_j = Q_{ki} Q_{lj} a_k b_l$ (double sum over k and l). Therefore in the indicial notation, a *tensor of second order* has 2 free indices (9 components), *e.g.* T_{ij}, that transform according to the rule $T'_{ij} = Q_{ki} Q_{lj} T_{kl}$. Tensors are usually denoted with a capital letter. Scalars and vectors can be called tensor of 0th and 1st order, respectively. This approach directly leads to an extension to tensor of third, fourth and higher order. A tensor of order n, as n free indices and 3^n components (in 3D space) that transform in a systematic way. For instance, C_{ijk} is a third order tensor whose components in the x' basis are $C'_{ijk} = Q_{li} Q_{mj} Q_{nk} C_{lmn}$. In summary, if $x'_i = Q_{ki} x_k$ then tensor components must obey the transformation rules

$$a_i \to a'_i = Q_{ki} a_k, \qquad T_{ij} \to T'_{ij} = Q_{ki} Q_{lj} T_{kl}, \qquad C_{ijk} \to C'_{ijk} = Q_{li} Q_{mj} Q_{nk} C_{lmn}. \tag{17}$$

This systematic and automatic generalization is useful in the continuum theory of elastic materials, for instance, where the stress tensor T_{ij} is related to the deformation tensor E_{kl} through a fourth order tensor in general: $T_{ij} = C_{ijkl} E_{kl}$.

Exercises and Applications

Assume that the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is orthogonal and right-handed. Coordinates are expressed in that basis unless otherwise noted.

1. What are the matrix representations of $\boldsymbol{e}_2\boldsymbol{e}_1$, $\boldsymbol{e}_3\boldsymbol{e}_2$ and $\boldsymbol{e}_1\boldsymbol{e}_3$?

2. If $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_j \boldsymbol{e}_j$, calculate $(\boldsymbol{ab}) \cdot \boldsymbol{e}_3$, $\boldsymbol{e}_2 \cdot (\boldsymbol{ab})$ and $(\boldsymbol{ab})^T \cdot \boldsymbol{e}_1$.

3. What are the transposes and the symmetric parts of $\boldsymbol{e}_1\boldsymbol{e}_2$ and $\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_2\boldsymbol{e}_2$?

4. Verify (16).

5. The angular momentum of N rigidly connected particles of mass m_α, $\alpha = 1, \ldots, N$, rotating about the origin is $\boldsymbol{L} = \sum_{\alpha=1}^{N} m_\alpha (\boldsymbol{r}_\alpha \times (\boldsymbol{\omega} \times \boldsymbol{r}_\alpha))$, where $\boldsymbol{r}_\alpha$ is the position vector of particles α and $\boldsymbol{\omega}$ is the rotation vector of the rigid system of particles. Write $\boldsymbol{L}$ as the dot product of a tensor with the rotation vector $\boldsymbol{\omega}$. That tensor is the *tensor of inertia*, $\boldsymbol{\mathcal{I}}$, find its antisymmetric part.

6. Any vector $\boldsymbol{a} = \boldsymbol{a}_\parallel + \boldsymbol{a}_\perp$ where $\boldsymbol{a}_\parallel$ is parallel and $\boldsymbol{a}_\perp$ perpendicular to a given normalized (*i.e.* unit) vector $\boldsymbol{n}$. Then $\boldsymbol{a}_\parallel \equiv \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{a}) = (\boldsymbol{nn}) \cdot \boldsymbol{a}$, $\forall \boldsymbol{a}$. Therefore the parallel projection tensor $\boldsymbol{P}_\parallel \equiv \boldsymbol{nn}$. Show that the perpendicular projection tensor is $\boldsymbol{P}_\perp = \boldsymbol{I} - \boldsymbol{nn}$. Sketch, sketch, sketch!!! don't just stick with algebra, visualize.

7. Show that the tensor that expresses reflection about the plane perpendicular to $\boldsymbol{n}$ is $\boldsymbol{H} = \boldsymbol{I} - 2\boldsymbol{nn}$. This is called a Householder tensor. Its generalizations to N dimensions is an important tool in linear algebra to obtain the QR decomposition of a matrix and other similar operations. Sketch and visualize!

8. Show that right-hand rotation by an angle φ about $\boldsymbol{n}$ of any vector $\boldsymbol{a}$ is given by

$$R(\boldsymbol{a}) = \cos\varphi\, \boldsymbol{a}_\perp + \sin\varphi(\boldsymbol{n}\times\boldsymbol{a}) + \boldsymbol{a}_\parallel.$$

Sketch and visualize! Using (16) and earlier exercises, find the tensor $\boldsymbol{R}_\varphi$ that expresses this rotation. Express the components R_{ij} of $\boldsymbol{R}_\varphi$ with respect to the orthogonal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ using the alternating (Levi-Civita) tensor ϵ_{ijk}. [Hint: To find $\boldsymbol{R}_\varphi$ in a coordinate-free form, use (16): $\boldsymbol{a}\times\boldsymbol{n} = (\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} - \hat{\boldsymbol{y}}\hat{\boldsymbol{x}})\cdot\boldsymbol{a}$, where $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ are *any* (!) vectors such that $\hat{\boldsymbol{x}}\times\hat{\boldsymbol{y}} = \boldsymbol{n}$ (hence $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{n}$ form a right handed orthogonal basis)].

9. What are the components of the vector $\boldsymbol{b}$ obtained by right-hand rotation of the vector (1,2,3) by an angle $\pi/3$ about the direction (4,1,2)?